Dynamics of the collapse of a Bose–Einstein condensate

M.B. Sobnack [1] and F.V. Kusmartsev

Department of Physics, Loughborough University, Loughborough, Leicestershire LE11 3TU, United Kingdom

Abstract

We investigate the dynamical stability of a Bose–Einstein condensate in a non–isotropic harmonic magnetic trap and present the conditions for the collapse of the system. We show that the condensate may exhibit a variety of exciting phenomena. The system has two modes of viobration, transverse and longitudinal. If the mean–field interaction between the atoms is attractive and if the energy of the system is negative the condensate droplet collapses. When the collapse takes there is an interesting interplay between the modes of vibration.

Key words: Bose–Einstein condensate; collapse; attractive interactions

Most of the physical processes in BEC's are adequately described by mean–field theory [1]. In the theory, the strength of the interactions depends on the atom density and on the s–wave scattering length a. a can be positive or negative, its sign (and magnitude) depending crucially on the details of the atom–atom potential. Positive and negative values of a correspond to an effective repulsion and attraction between the atoms respectively. a is negative for ^{7}Li and ^{85}Rb and positive for the other atomic gasses in which BEC has been observed.

In a spatially homogeneous gas, an attractive interaction leads to ordinary classical condensation into a liquid or solid, preventing BEC in the metastable region of the phase diagram. However, confinement in an atom trap produces stabilizing forces which may enable the formation of a metastable BEC: in a magnetic trap, the contraction of the BEC competes with the kinetic zero–point energy which tends to spread out the condensate. For a strong enough attractive interaction, there is not enough kinetic energy to stabilize the BEC and it is expected to implode. However, if the number of condensed atoms is less than some critical value N_m, the condensate is metastable. For a spherically symmetric harmonic trap $V(\mathbf{r}) = m\omega^2\mathbf{r}^2/2$, it can be shown [2] that in mean–field theory at $T = 0\,$K $N_m = 0.57l/|a|$, where $l = (\hbar/m\omega)^{1/2}$ is the extent of the one–particle ground state (i.e., the length scale of the condensate) in the harmonic trap. For $T > 0$, K the number is somewhat reduced. For the axially symmetric trap with ^{7}Li used in the experiments at Rice University [3], mean–field theory predicts $N_m \simeq 1400$ [4], consistent with experimental measurements.

The natural starting point for studying the behavior of these harmonically trapped BEC's is the theory of weakly interacting bosons which, for inhomogeneous systems, take the form of the Gross–Pitaevskii theory [5]. This mean–filed approach for the order parameter associated with the condensate is well–suited to describe most of the effects of two–body interactions in these dilute gasses.

When short–range correlations can be neglected and if the condensate wave function $\psi(\mathbf{r}, t)$ changes slowly on length scales of the order of the range of the interatomic potential, the dynamics of the wave function is well described by the nonlinear Schrödinger or Gross–Pitaevskii [5] equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + \beta|\psi|^2\right]\psi, \tag{1}$$

where m is the mass of the condensate atom, $V(\mathbf{r})$ is the external trapping potential and $\beta = 4\pi a\hbar^2/m$ is

[1] Corresponding author. E-mail: M.B.Sobnack@lboro.ac.uk

the two–body T matrix. We consider the anisotropic harmonic potential

$$V(\mathbf{r}) = V_\perp(\mathbf{R}) + V_\parallel(z) = \frac{1}{2}\omega_R \mathbf{R^2} + \frac{1}{2}\omega_z z^2, \qquad (2)$$

where $\mathbf{r} = (\mathbf{R}, z)$. Note that Eq. (1) ignores the mean–field contribution from the noncondensed atoms since it is nearly constant over the size of the condensate and hence only slightly affects the condensate dynamics. Below we put $\hbar = m = 1$.

Equation (1) has two integrals of motion: the total number of atoms in the droplet, $N = \int_{-\infty}^{\infty} d^3r\, |\psi(\mathbf{r},t)|^2$, and the total energy of the droplet,

$$E = \int\limits_{-\infty}^{\infty} d^3\mathbf{r}\, \left[\frac{1}{2}|\nabla\psi|^2 + V(r)|\psi|^2 + \frac{1}{2}\beta|\psi|^4\right]$$

$$= E_0 + \int\limits_{-\infty}^{\infty} d^3\mathbf{r}\, V(r)|\psi|^2, \qquad (3)$$

where $\nabla = \nabla_{\mathbf{R}} + \hat{\mathbf{e}}_z \partial/\partial z$. To investigate the collapse of the system, we consider the quantities

$$I_R(t) = \int d^3\mathbf{r}\, R^2 |\psi(\mathbf{r},t)|^2,$$

$$I_z(t) = \int d^3\mathbf{r}\, z^2 |\psi(\mathbf{r},t)|^2.$$

See also Refs. [6,7] Differentiating I_R and I_z with respect to t and integrating by parts gives

$$\frac{d^2 I_R}{dt^2} = -2\int d^3\mathbf{r}\, |\nabla\psi|^2 - 2\int d^3\mathbf{r}\, |\nabla_{\mathbf{R}}\psi|^2 - 2\int d^3\mathbf{r}\, \mathbf{R}\cdot\nabla_{\mathbf{R}} V_R |\psi|^2 + 4E_0 \qquad (4)$$

and

$$\frac{d^2 I_z}{dt^2} = -\int d^3\mathbf{r}\, |\nabla\psi|^2 + 2\int d^3\mathbf{r}\, \left|\frac{\partial\psi}{\partial z}\right|^2 - 2\int d^3\mathbf{r}\, z\frac{dV_z}{dz}|\psi|^2 + 2E_0. \qquad (5)$$

With the explicit form (2) for the confining potential $V(\mathbf{r})$, the above equations give the pair of coupled equations

$$\frac{d^2 I_R}{dt^2} = 4E - 4\omega_R^2 I_R - 2\omega_z^2 I_z - 2\int d^3\mathbf{r}\, \left|\frac{\partial\psi}{\partial z}\right|^2$$

$$\frac{d^2 I_z}{dt^2} = 2E - \omega_R^2 I_R - 3\omega_z^2 I_z + \int d^3\mathbf{r}\, \left|\frac{\partial\psi}{\partial z}\right|^2 - \int d^3\mathbf{r}\, |\nabla_{\mathbf{R}}\psi|^2. \qquad (6)$$

Note that Eqns. (6) are equally applicable for disk–shaped and cigar–shaped form of the condensate droplets.

The first of these equations satisfies the relation

$$\frac{d^2 I_R}{dt^2} < 4E,$$

which leads to the inequality

$$I_R(t) < 2Et^2 + C_1 t + C_2, \qquad (7)$$

where C_1 and C_2 are constants. If $E < 0$, then, since I_R is positive, the above inequality can be satisfied only for not too high values of t. This means that the solution of the initial problem with $E < 0$ exists only for a finite time and leads to a point or line singularity at a certain $t = t_0$. The line singularity may give rise to the formation of filaments.

If the right hand sides of (6) are positive, there are two normal modes of oscillations: transverse and longitudinal. For traditional cigar–shaped droplets [8], $\omega_R \gg \omega_z$, the frequency of these modes are $\omega_\perp \approx 2\omega_R$ and $\omega_\parallel \approx \sqrt{5/2}\,\omega_z$. These modes have been observed in [8]. When $E < 0$, the system undergoes point collapse. Depending on the duration time τ_c of the collapse, the amplitudes of these vibrations may increase ($\tau_c\omega \gg 1$) or decrease ($\tau_c\omega \ll 1$). Amplification of one of the modes may give rise to the burst focus or jet effect reported by Donley *et al.* [8], who observed a stream of atoms with highly anisotropic velocities emerging from the collapsing condensate.

Acknowledgements

M.B.S. would like to acknowledge financial support from the Science Faculty, Loughborough University.

References

[1] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, S. Stringari, Rev. Mod. Phys. **71** (1999) 463.

[2] P.A. Ruprecht, M. J. Holland, K. Burnett, Phys. Rev. A **51** (1995) 4704.

[3] C.C. Bradley, C.A. Sackett, J.J. Tollett, R.G. Hulet, Phys. Rev. Lett. **75** (1995) 1687; C.C. Bradley, C.A. Sackett, R.G. Hulet, Phys. Rev. Lett. **78** (1997) 985; C.A. Sackett, C.C. Bradley, M. Welling, R.G. Hulet, Appl. Phys. B **65** (1997) 433.

[4] F. Dalfovo, S. Stringari, Phys. Rev. A **53** (1996) 2477; R.J. Dodd, M. Edwards, C.J. Williams, C.W. Clark, M.J. Holland, P.A. Ruprecht, K. Burnett, Phys. Rev. A **54** (1996) 661.

[5] E.P. Gross, Nuove Cimento **20** (1961) 454; L.P. Pitaevskii, Sov. Phys. JETP **13** (1961) 451; E.P. Gross, J. Math. Phys. **4** (1963) 195.

[6] V.E. Zakharov, Sov. Phys. JETP **35** (1972) 908.

[7] L.P. Pitaevskii, Phys. Lett. A **221** (1996) 14.

[8] E.A. Donley *et al.*, Nature **412**(2001) 295.