Vortex nucleation, transition to turbulence, and cavitation: “system failure” experiments in liquid helium and extreme value statistics

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Abstract

It is suggested that recent experiments in liquid helium, like single vortex nucleation, transition to turbulent flow around a sphere at a critical velocity, and cavitation of the liquid in a sound wave belong to the type of “system failure” experiments which is well known in reliability testing and whose statistical properties are described by extreme value statistics. This leads to far reaching consequences for the interpretation of the critical velocities and of the voltage threshold for cavitation.

Key words: vortex nucleation; turbulence; cavitation; extreme value statistics

1. Introduction

The high-voltage breakdown of an insulator, the breaking strength of a piece of material, the lifetime of vacuum tubes or semiconductor devices, e.g., are typical *“system failure”* experiments in which a system (a device consisting of components, a block of material, etc.) is tested either for its lifetime under constant conditions of operation or by changing some external load (voltage, temperature, pressure, etc.) until it fails to function. In all these experiments the so-called *“weakest link”* principle is at work: the system fails when its weakest part fails. Similarly, when superfluid helium flows through an orifice or around a solid body, vortex lines will be generated first at that position on the solid surface where the barrier for nucleation (or the critical velocity) is minimal. And in experiments on cavitation, like in high-voltage breakdown, the threshold will be determined by the weakest nucleation site. Therefore, it is plausible to analyze these experiments in a similar way as the other failure experiments.

2. Extreme value statistics

The statistical analysis of an experiment in which the minimum (or maximum) of a sequence of independent random variables $x_i\,(i = 1, ..., n)$ having a common cumulative distribution function (CDF) $F(x)$, is measured (all other values x_i need not be observable) requires the mathematical tool of “extreme value theory” or “extreme value statistics” (EVS) [1]. It is a central result of EVS to derive the particular extreme value distribution (EVD) of the minimum in the limit of large sample size n. Interestingly, only three types of EVD exist. In standardized form these are, see Fig. 1:

Type 1: $H_1(x) = 1 - \exp(-\exp(x)), -\infty \leq x \leq \infty$, the “Gompertz” distribution;

Type 2: $H_2(x) = 1 - \exp(-(-x)^{-\alpha}),\ x < 0,\ \alpha > 0$, the “Fréchet” distribution;

Type 3: $H_3(x) = 1 - \exp(-x^{\alpha}),\ x \geq 0,\ \alpha > 0$, the “Weibull” distribution.

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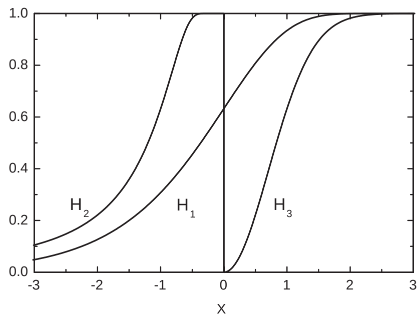

Fig. 1. The three extreme value distributions for minima. Note that H_2 and H_3, which are shown here for $\alpha = 2$, are limited to $x < 0$ and $x \geq 0$, respectively. For maxima three corresponding EVDs exist: $G(x) = 1 - H(-x)$.

In the single vortex nucleation experiments [2,3] the Gompertz distribution $H_1(x)$ describes the CDF of the nucleation probability as a function of the flow velocity. The validity of H_1 can be proven not only by a fit to the data but also by a unique property of H_1, namely an exponential "failure rate" (or "hazard function") $\Lambda_1(x)$ which is defined as [1]

$$\Lambda_1(x) \equiv \frac{H_1'(x)}{1 - H_1(x)} = -\frac{\mathrm{d}}{\mathrm{d}x} \ln(1 - H_1(x)) = \exp(x).$$

By numerical differentiation of the data of Refs. 2 and 3 the exponential failure rate and hence the validity of the Gompertz CDF was demonstrated [4]. Also in the cavitation experiment this EVD is observed for the excitation voltage for cavitation [5]. In the experiment on the transition to turbulence around an oscillating sphere, however, the Weibull distribution with $\alpha = 2$ (the "Rayleigh" distribution) determines the onset of turbulence as a function of the velocity increase above the critical velocity [6,7]. It is significant that this distribution is limited from below ($x \geq 0$): turbulence exists only above the critical velocity. The second type, the Fréchet distribution, is not observed in this context, probably because of its property of being limited from above. The very fact that two EVDs are observed in these three completely different experiments strongly supports our suggestion of considering these experiments as further examples of system failure.

The question of which initial CDF $F(x)$ ("parent distribution") of the n random variables leads to which type of EVD is answered by EVS. The set of different $F(x)$ which have the same EVD is called the "domain of attraction" of that particular EVD. The domain of attraction of the Gompertz CDF includes the following parent distributions: the normal, log-normal, logistic, Gumbel, and the Gompertz itself (each EVD belongs to its own domain of attraction). The Rayleigh distribution has the gamma, the log-logistic, and the Rayleigh distribution itself in its domain of attraction. Consequently, *it is impossible to determine the parent distribution* from the measured EVD, only the domain of attraction will be known, at least to some extent.

3. Data analysis

Suppose the measured CDF is the Gompertz as in vortex nucleation [2,3] and cavitation [5]. It is determined by the two parameters a_n and b_n:

$$H_1(x) = 1 - \exp\left(-\exp\left(a_n(x - b_n)\right)\right), \qquad (1)$$

a_n is the shape parameter which is related to the "width" of the CDF and b_n is the shift parameter which enters into the definition of a "critical" velocity or a "threshold" voltage, respectively, by the median of the CDF. *Both parameters depend not only on the particular parent distribution but also on the sample size n.* These functions can be found in the literature [1]. For example, if $F(x)$ is the (standardized) normal distribution the parameters are given by the following equations:

$$a_n = \sqrt{2\ \ln(n)}\ , \qquad b_n = -a_n + \frac{\ln(\ln(n)) + \ln(4\pi)}{2\, a_n}$$

In this particular case convergence is rather slow and hence large sample sizes n are necessary for a good fit of Eq. 1 to the data. Obviously, with unknown parent distribution and unknown sample size *no information can be inferred from the data* except for the statement that this is how the *minimum* is distributed. Even in the most favorable case when $F(x)$ is given by the (standardized) Gompertz CDF itself, we have [1]:

$$a_n = 1\,, \quad b_n = -\ln(n)\,,$$

i.e., the shape parameter is the same for both the parent distribution and the EVD but the shift parameter remains unknown as long as n is unknown [8].

In case of the transition to turbulence around the oscillating sphere [6,7] the situation with the observed Rayleigh distribution is as follows. The shape parameter is $a_n = n$ for the log-logistic and the Rayleigh, and $n/2$ for the gamma distribution while the shift parameter is $b_n = 0$ for all three parent distributions [1], i.e., we have:

$$H_3(x) = 1 - \exp\left(-a_n x^2\right). \qquad (2)$$

In the experimental data $(\Delta v/v_w)^2$ stands for $a_n x^2$, where $\Delta v = v - v_t$ is the exceedance of the velocity amplitude v over the velocity amplitude v_t at turbulence. Therefore, the "characteristic" velocity v_w is scaled down by $\sqrt{n}$, i.e., a large n leads to a small v_w.

From the data we have $v_w = 4.8\,\mathrm{mm/s}$, independent of temperature [6,7]. This small value seems to indicate a large (and temperature independent) n for this sphere (radius $124\,\mu$m). Clearly, also in this case a knowledge of n would be extremely important.

4. Conclusions

Two things are clearly necessary for a more detailed interpretation of the measured EVDs:
1. A physical model for the parent distribution $F(x)$ of the nucleation sites, barriers, or critical velocities. In case of vortex nucleation this will include information on the structure of the surface, e.g., the statistical distribution of its roughness [1,9]. How the roughness then affects the distribution of the nucleation barrier or the critical velocity is an open question. This problem appears to be a difficult one.
2. An information on the size effects in the measured EVDs. Obviously, a small orifice has fewer nucleation sites than a macroscopic body. Hence, the sample size n will depend on the geometry. These size effects are known in extreme value statistics, e.g., in the context of material strength and high-voltage breakdown [1]. But in the present experiments they have not yet been investigated because it requires a systematic variation of the geometry of the orifices or of the microsphere - a rather tedious task. As a numerical example, the surface area of the Berkeley orifice (circumference times membrane thickness) is 0.326 $\mu\mathrm{m}^2$[3] which for $n = 1000$ nucleation sites would imply a mean distance between the sites of 18 nm (for the Paris orifice the result is 46 nm), a number which is not unreasonable because on the scale of the small coherence length of superfluid helium-4 (0.2 nm) any real surface will be rough. Furthermore, it might well be that in case of the cavitation experiment n is a random number with its own discrete distribution function, a situation which requires special treatment in EVS [10].

In summary, without any knowledge of the parent distribution and of the sample size it is impossible to extract more information from the measured EVD than the distribution of the minimum of an unknown number of random variables having an unknown distribution function. A quantitative analysis of the measured shape and shift parameters is not possible.

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[8] Note that it is impossible to determine n from the measured parameters because actually also $F(x)$ contains unknown shape and shift parameters which leads to three unknowns in the two equations for the parameters of the EVD.

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