

# Effect of Hund coupling on one-dimensional spin-orbital model

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## Abstract

The one-dimensional spin-orbital model perturbed by Hund coupling is studied by renormalization group and bosonization methods. The Hund coupling breaks the  $SU(4)$  spin-orbital symmetry into  $SU(2)_{\text{spin}} \times U(1)_{\text{orbital}}$  at weak coupling fixed point. The one-loop renormalization group analysis shows that the Hund coupling is relevant irrespective of Coulomb repulsion. When Coulomb repulsion is larger than Hund coupling, the spin-orbital physics in strong coupling regime is described by  $SO(6)$  Gross-Neveu model, where the spin and orbital excitations are gapped. When the Hund coupling is much larger than the Coulomb repulsion, the strong coupling regime is described by the two coupled  $SU(2)_{\text{spin}} \times SU(2)_{\text{orbital}}$  Gross-Neveu model, where again the spin and orbital excitations are gapped.

*Key words:* Hund Coupling; spin-orbital model; renormalization group

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## 1. Introduction

The interplay of spin and orbital degrees of freedom plays an important role in diverse correlated electron systems.[1] Recently, the one-dimensional (1D) spin-orbital models have been studied intensively motivated by the discovery of the quasi-1D spin-gapped materials,  $\text{Na}_2\text{Ti}_2\text{Sb}_2\text{O}$  and  $\text{Na}_2\text{V}_2\text{O}_5$ .[2] These materials can be modeled by a quarter-filled two-band Hubbard model, [3–6] and in the limit of strong Coulomb repulsion, the model can be mapped to the following coupled spin-chain model

$$H = K \sum_i (x + \mathbf{S}_i \cdot \mathbf{S}_{i+1}) (y + \mathbf{T}_i \cdot \mathbf{T}_{i+1}), \quad (1)$$

where  $\mathbf{S}_i$  and  $\mathbf{T}_i$  are the  $SU(2)$  spin and orbital operators in fundamental representation at site  $i$ , respectively.  $K \sim 8t^2/U$  is a coupling constant, where  $t$  is band-width, and  $U$  is Coulomb repulsion. The  $U(1)$  charge excitations are gapped at quarter-filling if the Hubbard  $U$  exceeds a critical value.[7] Therefore, at

low energy, the non-trivial dynamics of the Hamiltonian (1) reside in spin-orbital sector. The Hamiltonian (1) has an obvious  $SU(2)_{\text{spin}} \times SU(2)_{\text{orbital}}$  symmetry for generic values of  $x$  and  $y$ . For  $(x, y) = (\frac{1}{4}, \frac{1}{4})$ , the  $SU(2)_{\text{spin}} \times SU(2)_{\text{orbital}}$  symmetry is enhanced to  $SU(4)$  symmetry. At the  $SU(4)$  symmetric point, the Hamiltonian (1) becomes critical, and it can be described by  $SU(4)$  level 1 ( $k=1$ ) Wess-Zumino-Witten (WZW) model.[4–6] Furthermore, the  $SU(4)$  symmetric Hamiltonian is integrable by Bethe ansatz method.[8] The Hamiltonian (1) for generic values of  $x$  and  $y$  can be most naturally studied as a perturbation with respect to the  $SU(4)$  symmetric Hamiltonian.[4,6] The notable features of the obtained phase diagram of (1) are the existence of *extended critical* region in the vicinity of the symmetric point  $(x, y) = (\frac{1}{4}, \frac{1}{4})$  and the existence of the *massive* phase with an approximate  $SO(6)$  symmetry with a dimerization of spin and orbital singlets. [4–6]

In this paper, we study the breaking of  $SU(4)$  spin-orbital symmetry by *Hund* coupling between two bands[3] at quarter filling. The Hund coupling is expected to be present in more realistic description of spin-orbital systems.[3] If the bandwidth  $t$  is larger

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than the Hubbard repulsion  $U$  and the Hund coupling  $J$  (the weak coupling case), the bosonization and the *perturbative* renormalization group (RG) method can be employed. By expressing Hamiltonian in terms of charge  $U(1)$  and spin-orbital  $SU(4)$  currents, the  $U(1)$  charge degrees of freedom are shown to decouple from the  $SU(4)$  spin-orbital degrees of freedom, and the  $SU(4)$  symmetry of spin-orbital degrees of freedom are shown to be broken into  $SU(2)_{\text{spin}} \times U(1)_{\text{orbital}}$ . The one-loop renormalization group equations (RGE) which is valid in the weak coupling case can be obtained using current algebra techniques.[9–14] The analysis of RGE indicates that the Hund coupling is relevant irrespective of Coulomb repulsion, and it drives system to strong coupling regimes. The nature of the strong coupling regimes crucially depends on the relative magnitude of the Coulomb repulsion and the Hund coupling. When the Coulomb repulsion is larger than the Hund coupling, the RG flows of coupling constants strongly indicate the restoration of  $SU(4) \sim SO(6)$  symmetry[4,13,15] at the strong coupling regime. It turns out that the strong coupling regime can be described by  $SO(6)$  Gross-Neveu (GN) model, where the spin and orbital excitations are gapped. The detailed investigations reveal that the  $SO(6)$  symmetric strong coupling regime belongs to the same universality class of massive phase found by Azaria *et al.* and Itoi *et al.* apart from some inessential factors.[4,6]

When the Hund coupling is much larger than the Coulomb repulsion, the full restoration of  $SO(6)$  symmetry does not occur, but the orbital  $U(1)_{\text{orbital}}$  symmetry is enhanced to  $SO(3)_{\text{orbital}}$ . The strong coupling regime is described by the two coupled  $SU(2)_{\text{spin}} \times SO(3)_{\text{orbital}}$  GN model, where the spin and orbital excitations are also gapped.

Even if the spin and orbital excitations are gapped in both cases, the properties of spin-spin and orbital-orbital correlation functions are very different from each other, which is essentially due to the differences in the excitation spectrum between  $SO(6)$  and  $SO(3)$  GN model.

The studies of gapless charge excitations, the strong coupling case, and the details of caculations will be published elsewhere.[16]

## 2. Model

The Hamiltonian is given by [3]

$$H = H_t + H_U + H_J,$$

$$H_t = \sum_{i\alpha\alpha'\sigma} \left( -t_{i,i+1}^{\alpha\alpha'} c_{i\alpha\sigma}^\dagger c_{i+1\alpha'\sigma} + \text{H.c.} \right),$$

$$H_U = \frac{U}{2} \sum_{i\alpha\alpha'\sigma\sigma'} \left[ n_{i\alpha\sigma} n_{i\alpha'\sigma'} (1 - \delta_{\alpha\alpha'} \delta_{\sigma\sigma'}) \right],$$

$$H_J = -2J \sum_i \mathbf{S}_{i1} \cdot \mathbf{S}_{i2}, \quad (2)$$

where  $c_{i\alpha\sigma}$  is the electron operator with orbital  $\alpha = (1, 2)$  and spin  $\sigma$  at  $i$ -th site.  $\mathbf{S}_{i\alpha}$  is the spin 1/2 operator of the  $\alpha$ -band at  $i$ -th site. The hopping is assumed to be diagonal in orbital space,  $t_{i,i+1}^{\alpha\alpha'} = t\delta_{\alpha\alpha'}$ .  $t, U, J$  are taken to be positive. At this point, define the spin and orbital operators.

$$\mathbf{S} = \sum_{\alpha\sigma'\sigma} c_{\sigma'\alpha}^\dagger \left[ \frac{\vec{\sigma}}{2} \right]_{\sigma'\sigma} c_{\alpha\sigma},$$

$$\mathbf{T} = \sum_{\alpha\alpha'\sigma} c_{\alpha'\sigma}^\dagger \left[ \frac{\vec{\tau}}{2} \right]_{\alpha'\alpha} c_{\alpha\sigma}, \quad (3)$$

where  $\vec{\sigma}, \vec{\tau}$  are the Pauli matrices acting on the spin, orbital space, respectively.  $H_t + H_U$  possesses  $U(4) = U(1)_{\text{charge}} \times SU(4)_{\text{spin-orbital}}$  symmetry. [3,4,6] The Hund coupling  $H_J$  breaks the  $SU(4)_{\text{spin-orbital}}$  symmetry explicitly.

At low energy, with the free electron spectrum linearized, the lattice electron operators  $c_{i\alpha\sigma}$  can be replaced by the continuum chiral fermions.

In terms of the chiral fermions  $\psi_{(\frac{R}{L})\alpha\sigma}$ ,  $H_t$  becomes

$$H_t = \frac{v_F}{i} \sum_{\alpha,\sigma} \int dx \left[ \psi_{R\alpha\sigma}^\dagger \frac{\partial \psi_{R\alpha\sigma}}{\partial x} - \psi_{L\alpha\sigma}^\dagger \frac{\partial \psi_{L\alpha\sigma}}{\partial x} \right], \quad (4)$$

where  $v_F = 2t \sin k_F a_0$ . Let us introduce the chiral  $U(1)$  charge and  $SU(4)$  spin-orbital currents.

$$J_{L/R} = \sum_{\alpha\sigma} \psi_{L/R,\alpha\sigma}^\dagger \psi_{L/R,\alpha\sigma},$$

$$J_{L/R}^A = \sum_{\alpha'\sigma'\alpha\sigma} \psi_{L/R,\alpha'\sigma'}^\dagger [M^A]_{\alpha\sigma}^{\alpha'\sigma'} \psi_{L/R,\alpha\sigma}, \quad (5)$$

where  $M^A$  [ $A = 1, \dots, 15$ ] are the generators of  $SU(4)$  Lie algebra.[6] The convenient explicit realizations of  $M^A$  are

$$\frac{1}{\sqrt{2}} \left( \frac{\sigma^a}{2} \right)_{\sigma}^{\sigma'} \delta_{\alpha}^{\alpha'}, \frac{1}{\sqrt{2}} \delta_{\sigma}^{\sigma'} \left( \frac{\tau^a}{2} \right)_{\alpha}^{\alpha'}, \sqrt{2} \left( \frac{\sigma^a}{2} \right)_{\sigma}^{\sigma'} \left( \frac{\tau^b}{2} \right)_{\alpha}^{\alpha'}, \quad (6)$$

where  $\sigma^a, \tau^b$ , ( $a, b = 1, 2, 3$ ) are the Pauli matrices acting on spin and orbital space, respectively. The  $SU(4)$  matrices satisfy the normalization  $\text{tr}(M^A M^B) = \frac{1}{2} \delta^{AB}$ . We will also designate the generators Eq. (6) with a pair of indices,  $(a, b) \neq (0, 0)$ ,  $(a, b = 0, 1, 2, 3)$  with an understanding that  $\sigma^0 = \tau^0 = \text{Id}_2$ . For an example, the first three generators of Eq. (6) can be alternatively expressed as  $M^{(a,0)}$ .

Using Abelian and non-Abelian bosonizations[9–11],  $H_t + H_U$  can be expressed as

$$\begin{aligned}
H_t + H_U &= H_{U(1)} \\
+ \int dx \sum_A &\left[ \frac{2\pi v_{so}}{5} (J_L^A J_L^A + J_R^A J_R^A) - 2Ua J_L^A J_R^A \right], \\
H_{U(1)} &= \frac{v_\rho}{2} \int dx \left[ \frac{1}{K_\rho} (\partial_x \phi_\rho)^2 + K_\rho \Pi_\rho^2 \right] + H_{\text{um}}, \quad (7)
\end{aligned}$$

where  $v_{so} = v_F(1 - \frac{Ua}{2\pi v_F})$  and  $H_{\text{um}} \propto U^2 \cos[4\sqrt{\pi}\phi_\rho]$ .  $H_{U(1)}$  is the Hamiltonian of charge degrees of freedom. The charge boson  $\phi_\rho$  and its conjugate momentum  $\Pi_\rho$  are related with  $U(1)$  currents through

$$\begin{aligned}
J_R(x) + J_L(x) &= \sqrt{\frac{4}{\pi}} \partial_x \phi_\rho(x), \\
J_R(x) - J_L(x) &= -\sqrt{\frac{4}{\pi}} \Pi_\rho(x), \\
v_\rho &= v_F \left[ (1 + \frac{2Ua}{\pi v_F})(1 - \frac{Ua}{\pi v_F}) \right]^{1/2} > v_F, \\
K_\rho &= \left[ (1 - \frac{Ua}{\pi v_F})/(1 + \frac{2Ua}{\pi v_F}) \right]^{1/2} < 1. \quad (8)
\end{aligned}$$

The scaling dimension of umklapp operator is  $4K_\rho$ . With large  $U$  such that  $4K_\rho < 2$  charge gap would open.

In the form of Eq. (7), the charge and spin/orbital degrees of freedom are explicitly separated on the Hamiltonian level. Employing the completeness relation of  $SU(N)$  Lie algebra,

$$\sum_a T_{lm}^a T_{pq}^a = \frac{1}{2} \left[ \delta_{lq} \delta_{mp} - \frac{1}{N} \delta_{lm} \delta_{pq} \right], \quad (9)$$

the total action of spin-orbital degrees of freedom becomes

$$S_{so} = S_{\text{WZW}}(SU(4)_{k=1}, v_{so}) \quad (10)$$

$$- \lambda_1 \int dx d\tau \sum_a \left[ J_L^{(a0)} J_L^{(a0)} + J_R^{(a0)} J_R^{(a0)} \right] \quad (11)$$

$$- \lambda_2 \int dx d\tau \sum_a \left[ J_L^{(a3)} J_L^{(a3)} + J_R^{(a3)} J_R^{(a3)} \right]$$

$$- \tilde{g}_1 \int dx d\tau \sum_a \left[ J_L^{(a0)} J_R^{(a0)} \right] \quad (12)$$

$$- \tilde{g}_2 \int dx d\tau \sum_a \left[ J_L^{(a,1)} J_R^{(a,1)} + J_L^{(a,2)} J_R^{(a,2)} \right] \quad (13)$$

$$- \tilde{g}_3 \int dx d\tau \sum_a \left[ J_L^{(a3)} J_R^{(a3)} \right]$$

$$- \tilde{g}_4 \int dx d\tau \left[ J_L^{(0,1)} J_R^{(0,1)} + J_L^{(0,2)} J_R^{(0,2)} \right] \quad (14)$$

$$- \tilde{g}_5 \int dx d\tau \left[ J_L^{(0,3)} J_R^{(0,3)} \right],$$

where  $S_{\text{WZW}}(SU(4)_{k=1}, v_{so})$  is the WZW action for  $SU(4)_{k=1}$  Kac-Moody algebra with the "speed of light" given by  $v_{so}$ . Initial values are given by

$$\begin{aligned}
\lambda_1(0) &= -\lambda_2(0) = Ja, \quad \tilde{g}_1(0) = 2Ua + 2Ja, \\
\tilde{g}_2(0) &= 2Ua + Ja, \quad \tilde{g}_3(0) = 2Ua - 2Ja, \\
\tilde{g}_4(0) &= 2Ua - 3Ja, \quad \tilde{g}_5(0) = 2Ua. \quad (15)
\end{aligned}$$

The symmetry of spin sector is clearly seen to be  $SU(2)$ , while the symmetry of orbital sector is  $U(1)$ . Namely only the rotation about the third axis in orbital space is a symmetry. The importance of the symmetry breaking Hund coupling at low energy can be assessed by RG flows of Hund coupling.  $\tilde{g}_1$  and  $(\tilde{g}_4, \tilde{g}_5)$  describe the interactions in the spin and orbital sector, respectively.  $\tilde{g}_2$  and  $\tilde{g}_3$  couple the spin and orbital degrees of freedom. If the orbital sector were  $SU(2)$  symmetric, then the equalities  $\tilde{g}_2 = \tilde{g}_3$  and  $\tilde{g}_4 = \tilde{g}_5$  would hold.

Using the current algebra technique[12–14], the one-loop RGE can be obtained. The one-loop RGE of current-current type interactions is essentially determined by the structure constants of underlying Lie algebra [12,13], which is  $SU(4)$  in our case. Since  $\lambda_1, \lambda_2$  terms of Eq. (11) are chiral, they are not renormalized in the leading order of  $U, J$ .[14] Thus, at one-loop level, we need only to consider the renormalizations of  $\tilde{g}_i$ 's. Define the dimensionless coupling constants,  $g_i = \frac{\tilde{g}_i}{4\pi v_{so}}$ , and let  $t = \ln L$  be the RG time, where  $L$  is the cut-off length scale. The one-loop RGE are

$$\begin{aligned}
\frac{dg_1}{dt} &= -g_1^2 - 2g_2^2 - g_3^2 \\
\frac{dg_2}{dt} &= -2g_1 g_2 - g_2 g_5 - g_3 g_4 \\
\frac{dg_3}{dt} &= -2g_1 g_3 - 2g_2 g_4 \\
\frac{dg_4}{dt} &= -3g_2 g_3 - g_4 g_5 \\
\frac{dg_5}{dt} &= -3g_2^2 - g_4^2. \quad (16)
\end{aligned}$$

Note that the Eq. (16) are invariant under  $Z_2$  transformation  $(g_2, g_3) \rightarrow -(g_2, g_3)$ .

The Eq. (16) are analyzed in the next section using numerical integration and linear stability analysis.[13]

### 3. RG flows

Given the initial values of  $\{g_i\}$ ,  $(i = 1, \dots, 5)$ , the RG flows of Eq. (16) are uniquely determined. The initial values of  $g_i$  are in turn determined by  $k_F, U/t < 1, J/t < 1$ . In principle, the derived RGE Eq. (16) is valid until  $\{\max g_i\} \sim O(1)$ . If all of coupling constants converge to finite values as  $t \rightarrow \infty$ , the initial

fixed point is stable. If any coupling constant diverges, the initial fixed point is destabilized along the direction of the diverge coupling constant, and the *asymptotic* behaviour of diverging trajectory can be determined by RGE.[14] Non-zero Hund coupling  $J$  introduces anisotropy among the initial values of coupling constants. The anisotropy of initial values changes the characters of RG flows qualitatively. For our system, the important parameter which influences the RG flow is the relative magnitude of Coulomb repulsion and Hund coupling. Clearly all coupling constants flow into the strong coupling regime. Moreover, the coupling constants seem to converge as they enter the strong coupling regime.[see Fig. 1] Such a convergence of cou-

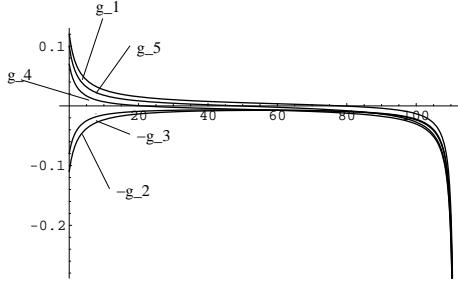


Fig. 1. The evolutions of  $g_1, g_4, g_5, -g_2, -g_3$  as a function of time for  $U/t = 0.5, J/t = 0.1$ .

pling constants *cannot* be an artifact of one-loop RGE since the convergence clearly *starts in the perturbative regime*,  $|g_i| < 1$ , where one-loop RGE is reliable. The RG flows of Fig. 1 clearly demonstrate that the set of coupling  $\{g_i\}$  flow onto fixed ratios, which is unity in our case, as they approach the strong coupling regime. The above convergence of coupling constants implies a restoration of  $SU(4)$  symmetry. But we have to keep in mind the sign reversal of  $g_2, g_3$ , and this point is discussed in the next section. The symmetry restoration as a system approaches (massive) strong coupling phase also occurs in the spin-orbital model considered by Azaria *et al.*[4] and in the  $SO(8)$  theory of two-leg Hubbard ladder at half-filling.[15] The above convergence of coupling constants can be understood by linear stability analysis.[13,16]

Now we turn to the case of  $J \gg U$ . Fig. 2 shows the evolutions of coupling constants for a case of  $J \gg U$ . Note the signs of  $g_2, g_3$  are *not* reversed contrary to the case of Fig. 1. Again all coupling constants flow into the strong coupling regime, but some new features emerge. Clearly,  $g_2$  and  $g_3$  merge in the *perturbative regime*, so do  $g_4$  and  $g_5$ . But the *total* RG flows do *not* converge to the isotropic ray in the *perturbative regime*. Indeed all flows tend to merge deep in the asymptotic region after  $g_1$  becomes negative, but then  $(g_2, g_3, g_4, g_5)$  lie too far outside the perturbative regime,[see Fig. 5] where one-loop RGE is not reliable. Thus, we can safely claim

that the RG flows approach the strong coupling regime with  $SU(2)_{\text{spin}} \times SU(2)_{\text{orbital}}$  symmetry, while a further enhancement of symmetry to  $SU(4)$  in the asymptotic region is questionable. The approach using the exact beta function indicates that the symmetry is *not* enhanced to  $SU(4)$  for the case of  $J \gg U$ .[16]

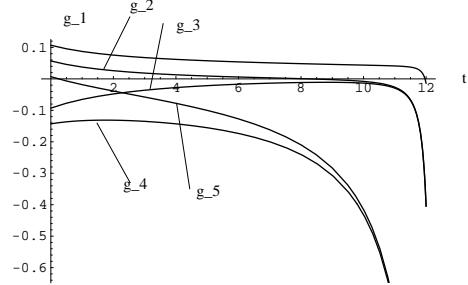


Fig. 2. The RG evolution of  $g_1, g_4, g_5, g_2, g_3$  as a function of time for  $U/t = 0.03, J/t = 0.5$ .

#### 4. Discussions and Summary

In discussing the symmetry breaking of  $SU(4)$  spin-orbital symmetry, it is convenient to use alternative bosonization of  $SU(4)_{k=1}$  Kac-Moody Hamiltonian. It is well-known[6,9,11] that  $SU(4)_{k=1}$  WZW model is equivalent to the sum of two decoupled  $SU(2)_{k=2}$  WZW model, where each  $SU(2)_{k=2}$  WZW model represents the spin and orbital degrees of freedom of our system. It is also well-known that  $SU(2)_{k=2}$  WZW model is equivalent to the triplet of massless Majorana fermions  $\xi_{s,\frac{R}{L}}^a, \xi_{t,\frac{R}{L}}^a$ .[18]  $\xi_{s,\frac{R}{L}}^a, \xi_{t,\frac{R}{L}}^a$  are the Majorana fermions associated with the spin / orbital  $SU(2)_{k=2}$  WZW model, respectively. The explicit expressions of  $SU(2)$  spin/orbital sub-currents of  $SU(4)$  currents in terms of Majorana fermions are given by [20]

$$\begin{aligned} J_R^{(a,0)} / \sqrt{2} &= -\frac{i}{2} \epsilon^{abc} \xi_{s,\frac{R}{L}}^b \xi_{s,\frac{R}{L}}^c, \\ J_R^{(0,a)} / \sqrt{2} &= -\frac{i}{2} \epsilon^{abc} \xi_{t,\frac{R}{L}}^b \xi_{t,\frac{R}{L}}^c. \end{aligned} \quad (17)$$

The  $g_2, g_3$  terms which couple the spin and orbital excitations can be expressed in terms of Majorana fermions as follows: (no summation over  $a$  and  $b$ )

$$J_R^{(a \neq 0, b \neq 0)} J_L^{(a \neq 0, b \neq 0)} = (\xi_{s,R}^a \xi_{s,L}^a) (\xi_{t,R}^b \xi_{t,L}^b). \quad (18)$$

For later conveniences introduce the notations  $\kappa_{s/t}^a \equiv \xi_{s/t,R}^a \xi_{s/t,L}^a$ . Now let us discuss the implications of RG flows for two cases  $U > J$  and  $U \ll J$  on the basis of Majorana fermions.

*Case of  $U > J$* - The sign reversal of  $g_2, g_3$  which was necessary for the restoration of  $SU(4)$  symmetry

in strong coupling regime in case of  $U > J$  can be implemented by  $\xi_{t,\frac{R}{L}}^a \rightarrow \pm \xi_{t,\frac{R}{L}}^a$ , which was also necessary for the Hamiltonian considered by Azaria *et al.*[4,5] to acquire  $SO(6) \sim SU(4)$  symmetry. Thus, the physics of *spin-orbital* degrees of freedom in this strong coupling regime is essentially identical with those of massive phase described by  $SO(6)$  Gross-Neveu (GN) model[21,22] as discussed by Azaria *et al.*[4] Azaria *et al.* characterized the ground state by the alternating expectation values of the spin and orbital dimerization operators:

$$\Delta_s = (-)^i \mathbf{S}_i \cdot \mathbf{S}_{i+1}, \quad \Delta_t = (-)^i \mathbf{T}_i \cdot \mathbf{T}_{i+1}. \quad (19)$$

In GN model, the chiral symmetry  $(\xi_R, \xi_L) \rightarrow (\xi_R, -\xi_L)$  is spontaneously broken, resulting in the ground states with positive *or* negative expectation values  $\langle \xi_R \xi_L \rangle = \langle \kappa \rangle$ . The topological excitations which connect these two degenerate ground states are called kinks and anti-kinks. The dimerization operators Eq. (19) reduce to

$$\Delta_s \sim \sum_{a=1,2,3} \kappa_s^a, \quad \Delta_t \sim \sum_{a=1,2,3} \kappa_t^a. \quad (20)$$

Then, the ground state can be characterized by

$$\langle \Delta_s \rangle = -\langle \Delta_t \rangle = \pm \Delta_0, \quad (21)$$

which indicates the alternating spin and orbital singlets.[4]

Next let us consider spin-spin correlation function. Owing to the  $SO(6)$  symmetry orbital-orbital correlation function gives the same result as spin-spin correlation function up to a trivial phase factor. In the infrared limit the contributions from the lightest excitations dominate the correlation function, which is fundamental Majorana fermions in the case of  $SO(6)$  GN model.[4,22] [More precisely, the mass of Majorana fermions is smaller than the twice the kink mass.]

$$\langle \mathbf{S}(x, \tau) \cdot \mathbf{S}(y, 0) \rangle \sim \cos(2k_F x) \cos(2k_F y) K_0(mR) - e^{4ik_F x} B^2 K_0^2(mR), \quad (22)$$

where  $R = \sqrt{(x-y)^2 + v_{so}^2 \tau^2}$  and  $K_0(mR)$  is the real space propagator of a free massive Majorana fermions. The first term of Eq. (22) would give rise to the *coherent* magnon peak at  $k = 2k_F$  and the second term represents the *incoherent* part at  $k \sim 4k_F$ .

*Case of  $J \gg U$* - For the case of  $J \gg U$ , in contrast to the case of  $J < U$ , the sign reversals of  $g_2$  and  $g_3$  in strong coupling regime do not occur. The symmetry in orbital sector is restored from  $U(1)$  to  $SU(2)$  even *before* entering the strong coupling regime as shown in Fig. 2. An interesting feature in the strong coupling regime is a hierarchy of coupling constants:

$$(|g_4| \sim |g_5|) \gg (|g_2| \sim |g_3|) \gg |g_1|. \quad (23)$$

Then the effective Hamiltonian in the strong coupling regime consists of two *coupled*  $SO(3)$  GN models, one in the spin sector and the other in orbital sector. Explicitly, [ $g_2 = g_3$  and  $g_4 = g_5$  are imposed]

$$\begin{aligned} H = & -i \frac{u_s}{2} \sum_a \left[ \xi_{sR}^a \partial_x \xi_{sR}^a - \xi_{sL}^a \partial_x \xi_{sL}^a \right] - g_1 \left( \sum_a \kappa_s^a \right)^2 \\ & - i \frac{u_t}{2} \sum_a \left[ \xi_{tR}^a \partial_x \xi_{tR}^a - \xi_{tL}^a \partial_x \xi_{tL}^a \right] - g_4 \left( \sum_a \kappa_t^a \right)^2 \\ & - g_2 \left( \sum_a \kappa_s^a \right) \left( \sum_a \kappa_t^a \right). \end{aligned} \quad (24)$$

Since  $|g_4|$  (remember  $g_4$  is negative) is much larger than other coupling constants, the orbital part [ the second line of Eq. (24) ] can be essentially treated separately in the leading approximation. Now for the (orbital)  $SO(3)$  GN model, the spontaneous breaking of chiral symmetry exists, leading to the finite expectation value of (orbital) dimerization operator  $\langle \sum_a \kappa_t^a \rangle = \Delta_t \neq 0$ . Thus, the dimerization in orbital sector is expected. However,  $SO(3)$  GN model does *not* possess elementary (Majorana) fermions in the excitation spectrum.[22,23] Only kinks and anti-kink remain in the spectrum. Following the discussions on the spin-spin correlation function in the previous section, we expect that the *coherent*  $2k_F$  peak would be *absent* in the orbital-orbital correlation function. This is because the kink excitations can not be built from the finite number of elementary (Majorana) fermions. Recall that the kinks change the sign of  $\langle \sum_a \kappa_t^a \rangle = \Delta_t$ .[22] The mean-field result suggests that[21] the gap of the orbital excitation is the order of  $\Delta_t \propto e^{-u_t/|g_4|}$ . Thus the expectation value of orbital dimerization operator becomes very large in the strong coupling regime. Once the very large condensate  $\langle \sum_a \kappa_t^a \rangle = \Delta_t$  in orbital sector is formed, the orbital sector can be safely integrated out, leaving us with [Recall  $g_2 < 0$ ]

$$\begin{aligned} H = & -i \frac{u_s}{2} \sum_a \left[ \xi_{sR}^a \partial_x \xi_{sR}^a - \xi_{sL}^a \partial_x \xi_{sL}^a \right] - g_1 \left( \sum_a \kappa_s^a \right)^2 \\ & - g_2 \left( \sum_a \kappa_s^a \right) \Delta_t, \end{aligned} \quad (25)$$

which is the sum of *three massive* Majorana fermions with weak marginal coupling  $g_1$ . The Hamiltonian Eq. (25) describes the *spin sector* in strong coupling regime. The mass of spin excitations is given by  $m_{\text{spin}} = |g_2| \Delta_t$ . The weak coupling  $g_1$  only slightly renormalizes the fermion mass  $m_{\text{spin}}$ , and it can be safely ignored. Therefore, the spin sector at low energy is described by the *triplet of free massive Majorana fermions* or equivalently (off-critical) Ising models. In terms of the spin variables  $\mathbf{S}$ , this implies the *massive*  $S=1$  excitations being consistent with the Haldane conjecture.[12] The expectation values of spin dimerization operators from

Eq.(25) is given by  $\langle \sum_a \kappa_s^a \rangle \sim |g_2| \Delta_t \neq \Delta_t$ , which obviously violates  $SO(6)$  symmetry. Next, we expect that the *coherent* magnon peak would exists at  $k = 2k_F$  for the spin-spin correlatin function in contrast to the case of orbital-orbital correlation function. This is because the Majorana fermions are the only excitations in the spin sector. But the peak is supposed to appear at  $\omega = m_{\text{spin}}$ , which is very large due to very large  $\Delta_t$ . Thus,  $2k_F$  component of spin-spin correlation function has a negligible effect to the low energy physics. The boundary between  $U > J$  and  $U \ll J$  is studied in [16].

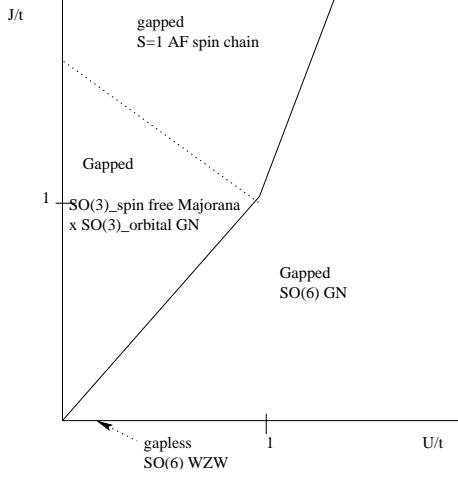


Fig. 3. The phase diagram based on the properties *spin-orbital* degrees of freedom *only*. The symmetry and the effective model at the strong coupling regime is indicated. Every boundary represents smooth crossover rather than critical quantum phase transitions.

**Summary-** We have studied the 1D spin-orbital model perturbed by Hund coupling  $J$ . Hund coupling turns out to be relevant irrespective of short range Coulomb repulsion  $U$ , which drives the system to strong coupling regime. When the Coulomb repulsion is larger than the Hund coupling, the spin-orbital degrees of freedom in the strong coupling regime are described by  $SO(6)$  GN model. When the Hund coupling is larger than the Coulomb repulsion the spin-orbital system is described by two coupled  $SO(3)$  GN model with a hierarchy among coupling constants for weak coupling case. It turns out that the spin sector can be reduced to the theory of a sum of *three free massive Majorana fermions*, while the orbital sector is essentially equivalent to  $SO(3)$  GN model. The exact RG approach indicates that the two cases  $U > J$  and  $J > U$  are smoothly connected to each other via crossover. The above results are summarized as a phase diagram in  $(U, J)$  plane in Fig. 3.

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