On Berry phase in Bloch states

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Abstract

We comment on the relation between Berry phase and quantized Hall conductivities for charge and spin in some Bloch states, such as Bloch electrons in the presence of electromagnetic fields and quasiparticles in the vortex states of superfluid ^{3}He. One can find out that the arguments presented here are closely related to the spontaneous polarization in crystalline dielectrics and the adiabatic pumping.

Key words: Berry phase, Bloch states, (spin) quantum Hall effect, Chern number, spontaneous polarization in crystalline dielectrics, adiabatic pumping

1. Introduction

Berry revealed that a geometrical phase (Berry phase) arises from the adiabatic process of a quantum mechanical system around a closed loop in the parameter space. In spite of the fact that it is a phase of the wave function, it could be related to physical effects and, in some cases, has a connection with topological numbers. Recently, the Berry phase is argued in the context of crystalline dielectrics by King-Smith and Vanderbilt, and Resta. The adiabatic change of the Kohm-Sham potential was considered. It was shown that the polarization change occurs spontaneously (i.e. the electric field is held to be zero) and it is written by the Berry phase. In this paper, we argue the system in the presence of electromagnetic field. It is renowned that the quantized Hall effect (QHE) occurs in such systems in two dimensions (2D) and also 3D. We define the electric polarization and calculate the polarization change under the adiabatic change of the vector potential. Then, we can find out the similarity to the spontaneous polarization and also the adiabatic pumping, which is originally argued by Thouless and discussed actively in the mesoscopic systems at present. We also show that a parallel discussion can be made for the vortex states of superfluid ^{3}He in 2D. The detailed discussions and references are written in Refs.[1].

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2. Polarization in Bloch electrons in the presence of electromagnetic fields

First, we argue 2D system. Consider the electrons under a periodic potential $U(\mathbf{r}) = U(\mathbf{r}+\mathbf{e}_x a) = U(\mathbf{r}+\mathbf{e}_y a)$, here we consider the square lattice for simplicity. We introduce uniform electromagnetic fields that is represented by a vector potential $\mathbf{A}(t,\mathbf{r}) = -\mathbf{E}t + \frac{1}{2}\mathbf{B}\times\mathbf{r}$. Here, the electric field is weak i.e. $|\mathbf{E}| << 1$ so that the vector potential changes adiabatically. We use the adiabatic approximation and we consider the eigenstates of the Hamiltonian $H(t)$ at fixed t. We adjust the flux ϕ through the unit cell of the crystal becomes rational i.e. $\phi = (p/q)(hc/e)$. One can take an enlarged primitive lattice vectors, such as $\mathbf{e}_x$ and $q\mathbf{e}_y$, and then, the system has symmetric under the magnetic translation in terms of an enlarged unit cell (the magnetic unit cell) $0 \le k_x < 2\pi/a, 0 \le k_y < 2\pi/qa$. The eigenfunctions of the system $\Phi_{\mathbf{k}}(t,\mathbf{r})$ is in the Bloch state $\Phi_{\mathbf{k}}(t,\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}u_{\mathbf{k}}(\mathbf{r})$, where $\mathbf{k}$ is on the magnetic Brillouin Zone (MBZ) and the function $u_{\mathbf{k}}(t, \mathbf{r})$ obey the generalized Bloch conditions. The band splitting occurs and one obtains q magnetic subbands. The "Hamiltonian" for $u_{\mathbf{k}}(t, \mathbf{r})$ is written $H_{\mathbf{k}}(t)$. Because of the gauge invariance, one can see that the adiabatic parameter $-\mathbf{E}t$ is on the MBZ. i.e. one may write $H_{\mathbf{k}}(t) = H_{\mathbf{k}-\mathbf{E}t}$. We should note that $H_{\mathbf{k}}(t)$ is compactified on the MBZ. So, we can introduce the periodicity T for time when $\mathbf{E}//\mathbf{G}$, where $\mathbf{G}$ is the reciprocal lattice vector for the enlarged Bravais lattice. For example, $T = h/eEa$ for $\mathbf{E}//\mathbf{e}_x$ and $T = h/eEqa$ for $\mathbf{E}//\mathbf{e}_y$. Then, the Berry phase arises when one solves the time-dependent Shrödinger equation by using the adiabatic approximation. We define the electric polarization $\mathbf{P}(t)$ and calculate its change per a cycle of the adiabatic process, $\Delta\mathbf{P} = \int_0^T dt\dot{\mathbf{P}}(t)$. One can show that $\dot{P(t)}$ is equivalent to the QH current when the Fermi level lies in the energy gap, i.e. $\dot{\mathbf{P}}(t) = \sigma_{xy}\mathbf{E} \times \mathbf{e}_z$. The Hall conductance σ_{xy} is written by the Chern number and is quantized in the multiple of e^2/h. It has been pointed out that the Chern number is written by the Berry phase, and then, $\Delta\mathbf{P}$ is closely related to the Berry phase. It is obtained by the adiabatic process and its direction is perpendicular to the electric field. Then, it is quite different with the usual induced polarization under an electric field. $\Delta\mathbf{P}$ is similar to the spontaneous polarization in crystalline dielectrics which is induced adiabatically change of the Kohm-Sham potential and written by the Berry phase. In the system, Hamiltonian $H_{\mathbf{k}}(t)$ changes ac-like, and the current is dc. $\Delta\mathbf{P}$ which corresponds to the charge transfer density per cycle does not depend on T and quantized. The fact is analogous to the adiabatic pumping in 1D.

The discussion can be extended to the 3D system. In 3D, the QHE also occurs in the "rational" magnetic field. Recent argument by Koshino et. al. pointed out that it may be possible to realize the 3D QHE in the magnetic field around 40 Tesla in the organic compounds $(\mathrm{TMTSF})_2\mathrm{X}$. As well as 2D, the polarization change per cycle is equivalent to the time integral of 3D quantized Hall current, whose conductivity is represented by the three sets of the Chern number. It was shown that the conductivity, and also the polarization change is written by the Berry phase.

3. Berry phase and spin quantum Hall effect in the vortex state of superfluid ^{3}He in 2D

A parallel discussion can be made in the vortex state of rotating superfluid ^{3}He in 2D. Superfluid in rotation is an direct analogy of the type II supercondcutors with infinite London penetration depth. In 2D, A-phase is stabilized by the boundary effect and spin currents are well defined. we introduce adiabatically changing vector potential that couples to spin current. This coupling is equivalent to the Zeeman coupling between spin and a magnetic field with weak and homogeneous gradient in the rotating frame. The system has periodicity in terms of an enlarged unit cell (*not* the unit cell of the vortex lattice. See, Fig. 1), which is the direct analogy of the magnetic unit cell. Then, quasiparticles are in the Bloch states. The adiabatic

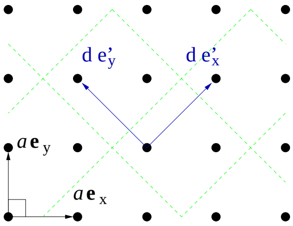

Fig. 1. The region surrounded by green dotted lines is the enlarged unit cell. Black dots shows vortices.

parameter is on the torus as well as the arguments in the previous section and the process can be closed. Then, a Berry phase is generated. When quasiparticles has an excitation gap, spin current flows perpendicular to the gradient field and quantized, i.e. spin quantum Hall effect (SQHE) occurs. The SQHE has been argued by Volovik and Yakovenko in ^{3}He-A without rotation, and Vishwanath, and Vafek et.al. in the vortex state of d-wave superconductors. The authors of the present paper have shown that the conductivity in the vortex state of superfluid ^{3}He is written by the Chern number as well as d-wave case, and moreover, closely related to the Berry phase. In the system, Hamiltonian changes ac-like and spin current is dc. The spin transfer ΔS_z per a cycle of the process per a unit cell is quantized and does not depends on the period of the cycle. Therefore, the effect is analogous to the adiabatic pumping. The magnetization change is defined as $\Delta S_z/d$ (d: the length of the boundary of the enlarged unit cell. See, Fig. 1). Obviously, it is written by the Berry phase and analogous to the spontaneous polarization in the crystalline dielectrics.

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