Spectral Statistics of a Spin-1/2 Particle in Coupled Quartic Oscillator Potentials

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Abstract

The quantum spectral statistical properties, i.e. the nearest neighbor level spacing distribution(NNSD) and the mode fluctuation distribution(MFD) of a spin-1/2 particle in three-dimensional coupled quartic oscillator potential are numerically studied. Introducing the coupling terms, the system can be continuously transformed from an integrable to chaotic one. With use of three parameter sets, three kinds of the Gaussian ensembles can be realized. The Brody-like NNSD interpolation formula between the Poisson distribution and that of each ensemble is found to work well for the numerically obtained NNSD.

Key words: quantum chaos, Gaussian ensembles, nearest neighbor level spacing distribution, mode fluctuation distribution

The study of quantum chaos has been spread out steadily from its early stage that concentrates on a spinless particle in two dimensions into the stage that provides the simplest and the most fundamental knowledge of quantum chaotic systems. However, in order to approach more realistic chaotic systems, e.g. mesoscopic systems, nano-technology etc., it is inevitable to to study the quantum motion of an electron, which is a spin-1/2 particle in three dimensions.

In this work, with different sets of parameters, our system has three kinds of chaotic limits, which correspond to the Gaussian Orthogonal Ensemble (GOE), the Gaussian Unitary Ensemble(GUE) and the Gaussian Symplectic Ensemble(GSE) [1], respectively. Those limiting cases have been studied both numerically and analytically using random matrix theory. However, the interpolation formula between the integrable and the GUE (or GSE) class has been scarcely studied. In this paper we also investigate the mode fluctuation distribution(MFD), which was introduced for another measure of chaoticity [2][3].

The Hamiltonian we choose is

$$\begin{aligned} H = &\frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + 3x^4 + 2y^4 + z^4 \\ &+ \lambda\left(x^2y^2 + y^2z^2 + z^2x^2\right) \\ &+ r^2\left(\alpha zx + \beta yz\right) + \gamma \mathbf{L}\cdot\mathbf{s} \end{aligned} \tag{1}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, $\mathbf{L}$ and $\mathbf{s}$ represent the radial distance from the origin, the orbital angular momentum and the spin operator(s=1/2). Note that we set dimensionless units: $\hbar = m = 1$, where m is the mass of the particle in the system. The last term represents the spin-orbit interaction. Changing coupling parameters : λ,α, β and γ, the system is continuously transformed from an integrable system to chaotic ones[4]. Putting all parameters zero, the system is just the superposition of three independent quartic oscillators and it is integrable. If these values are large enough to make the system chaotic, the choice of non-zero parameters α, β and γ makes the symmetry of the system different.

We put $\gamma > 0$ to emphasize the effect of the spin degree of freedom. However, all energy levels are still doublely degenerated. With $\alpha = 0$, $\beta = 0$ the system

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has the reflection symmetries with respect to the plane at $x = 0, y = 0$ and $z = 0$. The corresponding ensemble is the GOE in spite of introducing the spin. With $\alpha \neq 0$, $\beta = 0$ or $\alpha = 0$, $\beta \neq 0$ destroying part of the reflection symmetries, the ensemble becomes the GUE. With $\alpha \neq 0$, $\beta \neq 0$ the system conserves only the over-all parity, the ensemble finally becomes the GSE, due to the spin-orbit interaction.

The energy levels are obtained by numerical diagonalization of the truncated matrix of the Hamiltonian(1) in the basis of the harmonic oscillators[5] [6].We calculate 5000 eigen-energies out of about 23000 dimensional truncated matrices of H. Only 2500 eigen-energies are usuable, due to Kramers' degeneracy.

In chaotic regions we can derive very closely approximated function of NNSD which is derived from the correlated eigen-energy distribution function for two levels of the GOE, the GUE and the GSE[1]. For the GOE it is the Wigner distribution

$$P^{GOE}(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right). \quad (2)$$

For GUE we can obtain

$$P^{GUE}(s) = \frac{32}{\pi^2} s^2 \exp\left(-\frac{4}{\pi} s^2\right) \quad (3)$$

and for GSE

$$P^{GSE}(s) = \frac{2^{18}}{3^6 \pi^3} s^4 \exp\left(-\frac{64}{9\pi} s^2\right). \quad (4)$$

With the spin-orbit interaction, all energy levels are still doubly degenerated. Therefore we have to take every other levels to make the NNSD meaningful. At $\lambda = 1.4$ and $\gamma = 1.0$, we can get sufficiently chaotic systems and can confirm eq.(2)-(4) from the numerical levels. Putting $\alpha = \beta = 0$ it has linear repulsion near $s \approx 0$ and it is the Wigner distribution. With $\alpha \neq 0$ and $\beta = 0$, it presents the NNSD of GUE which has quadratic repulsion near $s \approx 0$. Finally the GSE class: $\alpha \neq 0$ and $\beta \neq 0$, it has quartic repulsion near $s \approx 0$.

In the intermediate region between integrable and chaotic ones, the most well known NNSD model is the Brody distribution[7]

$$P_q^{GOE}(s) = a_q s^q \exp\left(-b_q s^{1+q}\right), \quad (5)$$

where $a_q = (1+q)\, b_q$, $b_q = \left\{\Gamma\left(\frac{2+q}{1+q}\right)\right\}^{1+q}$, and $\Gamma(x)$ denotes the gamma function. At $q = 0$ it becomes the Poisson distribution $P(s) = exp(-s)$ and at $q = 1$ the Wigner distribution. The Brody distribution is especially the interpolation formula with no physical meaning, however, it has been applied to various systems and has been proven to work well. Following the Brody prescription, the interpolation formula for the GUE can be obtained

$$P_q^{GUE}(s) = a_q s^{2q} \exp\left(-b_q s^{1+q}\right), \quad (6)$$

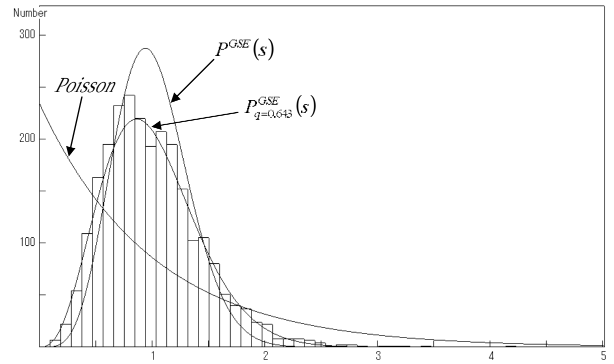

Fig. 1. Fitting result of eq.(7) for the system $\alpha = \beta = 0.1$, $\gamma = 0.03$ and $\lambda = 1.4$.

where $a_q = (1+q)\, b_q^2$, $b_q = \left\{\Gamma\left(\frac{1+2q}{1+q}\right)\right\}^{-(1+q)}$, and for the GSE

$$P_q^{GSE}(s) = a_q s^{4q} \exp\left(-b_q s^{1+q}\right) \quad (7)$$

where $a_q = (1+q)\,\Gamma\left(\frac{2+4q}{1+q}\right)^{1+4q} \Big/ \Gamma\left(\frac{1+4q}{1+q}\right)^{2+4q}$, $b_q = \left\{\Gamma\left(\frac{2+4q}{1+q}\right)/\Gamma\left(\frac{1+4q}{1+q}\right)\right\}^{1+q}$. Both $P_q^{GUE}(s)$ and $P_q^{GSE}(s)$ becomes Poissonian at $q = 0$. At q=1, of course, we can get eq.(3) from eq.(6) and eq.(4) from eq.(7). These interpolation formulae are also found to work well in the intermediate region. Fig.1 is the case in between the integrable and the GSE class, and the fitted curve is eq.(8) at $q = 0.643$. In addition, even at $\lambda = 0.0$ we can make the system chaotic to introduce the spin-orbit interaction.

The MFD is also tested for the GUE and the GSE system. In terms of the random matrix theory it is proven that the MFD becomes Gaussian in the limit of infinite rank[8]. In our chaotic limits the MFD always becomes Gaussian.

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