Geometry of fluctuating vortex loops at superfluid phase transitions

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Abstract

The geometrical properties of thermally-excited vortex loops near a superfluid phase transition are deduced from an analytic vortex-renormalization theory. The fractal Hausdorff dimension of the loops is $D_H = 2.5$, and the corresponding 'anomalous' dimensionality exponent is η_ϕ = -0.5. As the temperature is increased towards T_c the density distribution of loops of average diameter a crosses over from exponential to algebraic decay in the loop diameter. Just at T_c the distribution falls off algebraically as $a^{-\lambda}$, where λ = D+1 = 4.0, in exact agreement with a cosmic-string prediction of Vachaspati and Vilenkin.

Key words: superfluidity;phase transition;vortex loops;

The initial proposal that percolating vortex loops are the relevant thermal excitations of superfluid phase transitions [1] has now been verified in detail by Monte Carlo simulations [2]. An analytic theory employing vortex-loop renormalization methods has been formulated that provides an accurate description of the superfluid transition [3–6]. We examine here the geometry of the thermally-excited loops in this theory, and show that the loop diameter distribution is in exact agreement with a cosmic-string proposal of Vachaspati and Vilenkin [7].

1. Loop theory

The loop theory is quite simple, consisting of a recursion relation for the renormalized superfluid density $K_r = \hbar^2 \rho_s a_o / m^2 k_B T$ where m is the helium atom mass and a_o the bare core diameter,

$$\frac{1}{K_r} = \frac{1}{K_o} + \frac{4\pi^3}{3} \int_{a_o}^{a} \left(\frac{a}{a_o}\right)^{2D} \exp(-U(a)/k_B T) \, \frac{da}{a_o} \quad (1)$$

Here we have generalized the similar Eq. 1 of Ref. [6] to arbitrary dimension D, where a is the average loop diameter and K_o the initial bare superfluid density at the scale a_o. The renormalized loop energy is given by a second recursion relation

$$U(a)/k_B T = \pi^2 \int_{a_o}^{a} K_r (\ln\left(\frac{a}{a_c}\right) + 1) \frac{da}{a_o} + \pi^2 K_o C \quad (2)$$

where C is a nonuniversal constant. For the superfluid λ-transition it is found [5] that C=1.03 and a_o=2.53 Å. The effective core size a_c was found in a Flory-scaling entropy-energy minimization calculation [8] to be

$$\frac{a_c}{a} = \left(K_r \frac{a}{a_o}\right)^{\theta} \quad , \quad \theta = \frac{D}{D+2} \quad (3)$$

and from the same calculation the total perimeter of a loop of average diameter a is given by

$$\frac{p}{a_o} = B \left(\frac{a}{a_o}\right)^{1/\delta} \quad , \quad \delta = 1 - \theta = \frac{2}{D+2} \quad (4)$$

where the B is a constant of order unity [6]. The first equality in Eq. 4 has now been confirmed from general considerations of the random walk of a loop [9,10], defining the Hausdorff fractal dimension of the walk, $D_H = 1/\delta$. For D = 3 the Flory-scaling result is $\delta = 0.4$

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and hence $D_H = 2.5$. Associated with this fractal dimension is the "anomalous" dimensionality exponent [9] $\eta_\phi = D_H - D = -0.5$. Previous estimates of this exponent have ranged from -0.2 [11] to -0.38 [9] to -0.79 [12].

2. Loop diameter distribution

The number of loops per unit volume with diameter between a and $a+da$ is given by [6]

$$\frac{D(a)\,da}{L^3\,a_o} = \frac{\pi}{2\,a_o^3}\left(\frac{a}{a_o}\right)^{D-1}\exp\left(-\frac{U(a)}{k_BT}\right)\frac{da}{a_o} \qquad (5)$$

where L is the system size. At low temperatures $U \sim a$ and the distribution falls off exponentially in a. As T_c is approached however this changes to an algebraic decay in a. This is a consequence of a universality condition in the solutions of Eqs. 1 and 2 which is the 3D equivalent of the universal jump of the superfluid density in the 2D Kosterlitz-Thouless theory: at T_c the scale-dependent superfluid density varies as

$$K_r = D_o\left(\frac{a_o}{a}\right) \qquad (6)$$

where $D_o = 0.3875...$ is a universal constant [4,6]. Differentiating Eq. 1 and inserting the above relation for K_r gives

$$\exp(-U(a)/k_BT) = \frac{3a_o}{4\pi^3}\left(\frac{a}{a_o}\right)^{-2D}\frac{\partial}{\partial a}\left(\frac{1}{K_r}\right) \qquad (7)$$

$$= \frac{3}{4\pi^3 D_o}\left(\frac{a}{a_o}\right)^{-2D} \quad . \qquad (8)$$

Equation 5 then reduces to algebraic decay,

$$\frac{D(a)}{(L/a_o)^3} = E_o\left(\frac{a}{a_o}\right)^{-\lambda} \qquad (9)$$

where the exponent $\lambda = D+1 = 4.0$. This is exactly the prediction of Vachaspati and Vilenkin [7] for cosmic loops at T_c, which is based on the concept of scale invariance at the transition point. The prefactor $E_o = 3/(8\pi^2 D_o)$ is a universal constant. Figure 1 displays numerical solutions of Eqs. 1, 2, and 5 at various temperatures showing the details of the crossover from exponential to algebraic decay at the transition as Eq. 6 remains valid to longer and longer length scales.

Monte Carlo simulations for the distribution of loop diameters have recently been carried out [13] which also observe the crossover from exponential to algebraic decay. However, the exponent at T_c was found to be $\lambda = 4.16$, significantly higher than the exact value of 4.0. Olsson [12] has noted a problem that arises from the finite lattice used in the simulations, which cannot

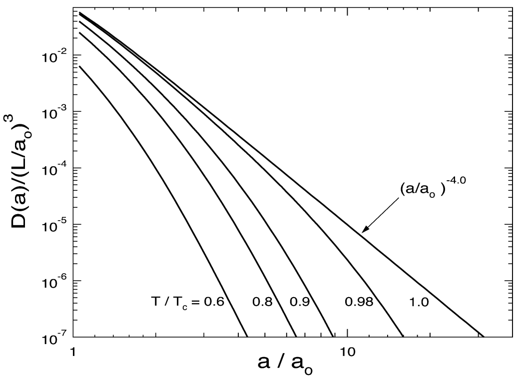

Fig. 1. Distribution of vortex loops as a function of their average diameter for several temperatures approaching T_c.

distinguish two loops approaching closer than a lattice constant from a single larger loop. This skews the distribution, giving rise to the higher exponent.

A similar crossover from exponential to algebraic decay was calculated [6] for the distribution of loops with total perimeter length p, where it was found at T_c that $D(p) \sim (p/a_o)^{-\gamma}$, with $\gamma = D\delta+1$ (Ref. [6] assumed D = 3). This relation between γ and δ has recently been confirmed in Ref. [9]. Values of δ can thus be extracted from the simulation distributions: $\delta = 0.41$ [14] and $\delta = 0.43$ [9]. Since as shown above these will be slightly too high due to the skewed distributions, they confirm very well the Flory-scaling result of Eq. 4.

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