The Self-Consistent Perturbation Theory of An Inhomogeneous Superconductor

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Abstract

For the purpose of investigating an inhomogeneous superconducting system, a theory of diordered superconductors subject to a magnetic field is evolved. In order to take account of the inhomogeneities induced in this superconducting system, we describe the generalized Hartree-Fock scheme to be written by the temperature Green's functions in the spatial representation, and reformulate the self-consistent perturbation theory of the Nambu formalism. And we evolve the gradient expansion theory on the basis of the Nambu formalism.

Key words: inhomogeneous superconductor; Nambu formalism; gradient expansion

1. Introduction

Recently, the possibility has been pointed out that increasing the disorder of an ultrathin superconducting film might produce a quantum phase transition to insulating state. The subject of the superconductor-insulator transition in two dimensions is an active area of both experimental and theoretical studies [1]. For the purpose of investigating such an inhomogeneous superconducting system, a theory of diordered superconductors subject to a magnetic field is evolved.

A self-consistent perturbation theory of superconductivity was evolved by Nambu [2] that the generalized Hartree-Fock scheme (i.e., the BCS theory) could be described as its first-order self-consistent perturbation theory. In order to take account of inhomogeneities induced in the superconducting system in magnetic field, we describe the generalized Hartree-Fock scheme to be written by the temperature Green's functions in the spatial representation, and reformulate the self-consistent perturbation theory of the Nambu formalism.

The problem of incorporating many-body effects outside the Hartree-Fock scheme into the inhomogeneous electron system was studied by Baraff et al [3]. Their gradient expansion method to expand termwise in powers of the gradient which measures the inhomogeneities was applied to the problem superconductivity by Werthamer et al [4]. For the inhomogeneous superconductor, we evolve the gradient expansion theory on the basis of the Nambu formalism of the self-consistent perturbation theory.

2. A self-consistent perturbation formulation

Such an inhomogeneous system is realized under the application of external magnetic field, or on the inhomogeneous lattice subjected to impurity potentials $U_{\rm imp}(\boldsymbol{r})$. When we introduce the Hartree-Fock spin-independent potential $U_{\rm HF}(\boldsymbol{r})$, and the gap functions $\boldsymbol{\Delta}(\boldsymbol{r}), \boldsymbol{\Delta}^*(\boldsymbol{r})$ which are caused by the BCS pairing interaction of a nonretarded zero-range attractive potential, we rewrite the grand canonical hamiltonian as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}' \qquad (1)$$

where $\mathcal{H}_0$ is given by

$$\mathcal{H}_0 = \int d\boldsymbol{r} \sum_\sigma \psi^\dagger_\sigma(\boldsymbol{r})\mathcal{D}(\boldsymbol{r})\psi_\sigma(\boldsymbol{r})$$

$$-\int d\boldsymbol{r}\Big\{\boldsymbol{\Delta}^*(\boldsymbol{r})\psi_\uparrow(\boldsymbol{r})\psi_\downarrow(\boldsymbol{r})+\boldsymbol{\Delta}(\boldsymbol{r})\psi_\downarrow^\dagger(\boldsymbol{r})\psi_\uparrow^\dagger(\boldsymbol{r})\Big\}, \quad (2)$$

with

$$\mathcal{D}(\boldsymbol{r})=\left\{\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial\boldsymbol{r}}+\frac{e}{c}\boldsymbol{A}(\boldsymbol{r})\right)^2-\mu\right\} +U_{\rm imp}(\boldsymbol{r})+U_{\rm HF}(\boldsymbol{r}), \quad (3)$$

and $\mathcal{H}'$ is given by

$$\mathcal{H}'=-g\int d\boldsymbol{r}\psi_\downarrow^\dagger(\boldsymbol{r})\psi_\uparrow^\dagger(\boldsymbol{r})\psi_\uparrow(\boldsymbol{r})\psi_\downarrow(\boldsymbol{r}) -\int d\boldsymbol{r}\Big\{\sum_\sigma\psi_\sigma^\dagger(\boldsymbol{r})U_{\rm HF}(\boldsymbol{r})\psi_\sigma(\boldsymbol{r}) -\boldsymbol{\Delta}^*(\boldsymbol{r})\psi_\uparrow(\boldsymbol{r})\psi_\downarrow(\boldsymbol{r})-\boldsymbol{\Delta}(\boldsymbol{r})\psi_\downarrow^\dagger(\boldsymbol{r})\psi_\uparrow^\dagger(\boldsymbol{r})\Big\}. \quad (4)$$

Here we assume that the Hartree-Fock potential $U_{\rm HF}(\boldsymbol{r})$, and the gap functions $\boldsymbol{\Delta}(\boldsymbol{r}),\boldsymbol{\Delta}^*(\boldsymbol{r})$ are to be determined self-consistently.

We can diagonalize $\mathcal{H}_0$ by using the generalized Bogoliubov transformatin to be defined by a set of orthonormal solutions which satisfy the Bogoliubov equation

$$\Big(\mathcal{D}(\boldsymbol{r})-E_\nu\Big)u_\nu(\boldsymbol{r})=-\boldsymbol{\Delta}(\boldsymbol{r})v_\nu(\boldsymbol{r}), \quad (5)$$

$$\Big(\mathcal{D}^*(\boldsymbol{r})+E_\nu\Big)v_\nu(\boldsymbol{r})=\boldsymbol{\Delta}^*(\boldsymbol{r})u_\nu(\boldsymbol{r}). \quad (6)$$

Under $\mathcal{H}_0$, the normal and superconducting temperature Green's functions can be explicitly expressed by

$$G_{0\uparrow\uparrow}(\boldsymbol{r},\boldsymbol{r}';i\omega_n)=G_{0\downarrow\downarrow}(\boldsymbol{r},\boldsymbol{r}';i\omega_n)\equiv G_0(\boldsymbol{r},\boldsymbol{r}';i\omega_n) =\sum_\nu\left\{\frac{u_\nu(\boldsymbol{r})u_\nu^*(\boldsymbol{r}')}{i\omega_n-(E_\nu/\hbar)}+\frac{v_\nu^*(\boldsymbol{r})v_\nu(\boldsymbol{r}')}{i\omega_n+(E_\nu/\hbar)}\right\}, \quad (7)$$

$$F_{0\uparrow\downarrow}(\boldsymbol{r},\boldsymbol{r}';i\omega_n)=-F_{0\downarrow\uparrow}(\boldsymbol{r},\boldsymbol{r}';i\omega_n)\equiv F_0(\boldsymbol{r},\boldsymbol{r}';i\omega_n) =\sum_\nu\left\{\frac{u_\nu(\boldsymbol{r})v_\nu^*(\boldsymbol{r}')}{i\omega_n-(E_\nu/\hbar)}-\frac{v_\nu^*(\boldsymbol{r})u_\nu(\boldsymbol{r}')}{i\omega_n+(E_\nu/\hbar)}\right\}. \quad (8)$$

The 2×2 Green's function matrix $\hat{\mathcal{G}}_0(\boldsymbol{r},\boldsymbol{r}';i\omega_n)$ defined by the G_0,F_0 and their complex conjugates satisfy a matrix equation

$$\left(i\hbar\omega_n\hat{1}-\hat{\mathcal{L}}(\boldsymbol{r})\right)\hat{\mathcal{G}}_0(\boldsymbol{r},\boldsymbol{r}';i\omega_n)=\hbar\delta(\boldsymbol{r}-\boldsymbol{r}')\hat{1}, \quad (9)$$

with

$$\hat{\mathcal{L}}(\boldsymbol{r})=\begin{pmatrix}\mathcal{D}(\boldsymbol{r}) & \boldsymbol{\Delta}(\boldsymbol{r})\\ \boldsymbol{\Delta}^*(\boldsymbol{r}) & -\mathcal{D}^*(\boldsymbol{r})\end{pmatrix}. \quad (10)$$

Under the full grand canonical hamiltonian $\mathcal{H}$, the corresponding Green's function matrix $\hat{\mathcal{G}}(\boldsymbol{r},\boldsymbol{r}';i\omega_n)$ satsifies the following matrix integral equation of motion

$$\hat{\mathcal{G}}(\boldsymbol{r},\boldsymbol{r}';i\omega_n)=\hat{\mathcal{G}}_0(\boldsymbol{r},\boldsymbol{r}';i\omega_n) +\int d\boldsymbol{r}_1\hat{\mathcal{G}}_0(\boldsymbol{r},\boldsymbol{r}_1;i\omega_n)\hat{\Sigma}(\boldsymbol{r}_1)\hat{\mathcal{G}}(\boldsymbol{r}_1,\boldsymbol{r}';i\omega_n). \quad (11)$$

Within the approximation which takes account of only the lowest-order self-energy cotribution, we require that the quasi-particle energy E_ν is unaffected and the self-consistent condition $\hat{\Sigma}\approx\hat{\Sigma}^{(1)}=\hat{0}$ is to be satisfied. Consequently, the self-consistent equations determining $U_{\rm HF}(\boldsymbol{r}),\boldsymbol{\Delta}(\boldsymbol{r})$, and $\boldsymbol{\Delta}^*(\boldsymbol{r})$ are derived as

$$U_{\rm HF}(\boldsymbol{r})=g\sum_\nu\Big\{u_\nu^*(\boldsymbol{r})u_\nu(\boldsymbol{r})\left(1-f(E_\nu)\right) +v_\nu^*(\boldsymbol{r})v_\nu(\boldsymbol{r})f(E_\nu)\Big\}, \quad (12)$$

$$\boldsymbol{\Delta}(\boldsymbol{r})=g\sum_\nu u_\nu(\boldsymbol{r})v_\nu^*(\boldsymbol{r})\left(1-2f(E_\nu)\right), \quad (13)$$

$$\boldsymbol{\Delta}^*(\boldsymbol{r})=g\sum_\nu u_\nu^*(\boldsymbol{r})v_\nu(\boldsymbol{r})\left(1-2f(E_\nu)\right), \quad (14)$$

where $f(E_\nu)=1/(e^{\beta E_\nu}+1)$ denotes a fermi distribution function for the quasi-particle energy E_ν.

In correspondence to the matrix equation (11) of motion, let's introduce the kernel $\hat{K}(\boldsymbol{r},\boldsymbol{s};i\omega_n)$ of a 2×2 matrix form to be defined by

$$\hat{K}(\boldsymbol{r},\boldsymbol{s};i\omega_n) \equiv\delta(\boldsymbol{r}-\boldsymbol{s})\left\{i\hbar\omega_n\hat{1}-\hat{\mathcal{L}}(\boldsymbol{r})\right\}-\hbar\hat{\Sigma}(\boldsymbol{r},\boldsymbol{s};i\omega_n). \quad (15)$$

Then, the matrix integral equation of motion for the Green's function matrix $\hat{\mathcal{G}}(\boldsymbol{r},\boldsymbol{r}';i\omega_n)$ is given by

$$\int d\boldsymbol{s}\hat{K}(\boldsymbol{r},\boldsymbol{s};i\omega_n)\hat{\mathcal{G}}(\boldsymbol{s},\boldsymbol{r}';i\omega_n)=\hbar\delta(\boldsymbol{r}-\boldsymbol{r}')\hat{1}. \quad (16)$$

By introducing center of mass coordinate $\boldsymbol{R}$ and relative coordinate $\boldsymbol{l}$ for the Green's functions, we evolve the gradient expansion theory due to inhomogeneities. As a result, we get the Green's function matrix $\tilde{\mathcal{G}}(\boldsymbol{k},\boldsymbol{R};i\omega_n)$ to be calculated up to the lowest-order of the gradient expasion. And so the self-consistent equations corresponding to $\hat{\Sigma}=\hat{0}$, which determine the self-consistent fields $U_{\rm HF}(\boldsymbol{R}),\boldsymbol{\Delta}(\boldsymbol{R})$ and $\boldsymbol{\Delta}^*(\boldsymbol{R})$, can be obtained.

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