Statistical mechanics of the gas-liquid condensation in the attractive Bose gas

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Abstract

The Bose gas with the attractive interaction is treated on the basis of the statistical mechanics. It is proved that, in cooling and compressing the system, the gas-liquid condensation (GLC) always occurs before the occurrence of the BEC. The critical size-distribution of the many-body wave function at the GLC point is obtained.

Key words: Bose statistics, gas-liquid transition, helium4, trapped atomic gas

1. Introduction

The gas-liquid condensation (GLC) is a universal phenomenon occurring both in the classical and quantum world. When the GLC occurs at high temperature, its main reason is the strong attractive interaction so that the quantum statistics plays a minor role. This is the classical GLC which we experience in the daily life.

When the GLC occurs at low temperature, however, its main reason is the weak attractive interaction so that the quantum statistics plays an important role. From now, we will call it the *quantum GLC* . The GLC in the attractive Bose gas at low temperature and high density is a prototype of the quantum GLC. (If the Bose particles form the Bose-Einstein condensation (BEC), their thermal equilibrium state is the liquid. The BEC gas is unstable to the attractive interaction. Hence, the GLC does not exist in the BEC phase.)

The region in which the quantum GLC is realized in the phase diagram is a normal phase in the vicinity of the BEC transition point. In this region, the many-body wave function, in which the permutation symmetry is satisfied, involves many Bose particles, but it does not reach the macroscopic size. If the weak attractive force acts on the Bose particles belonging to such a wave function, they abruptly undergo the GLC which is enhanced by the Bose statistics.

As an example of such a quantum GLC, we know two probable candidates. The first is the helium 4 gas at an extremely low temperature and an extremely low pressure. The second is the trapped atomic gas just above the BEC transition point.

To formulate the quantum GLC, we must deal with the attractive interaction between the Bose particles in such a way that the Bose statistics is exactly satisfied. This problem reminds us of the famous argument by Feynman, in which he describes the behavior of the Bose particles at low temperature by emphasizing the role of the many-body wave function [1]. By applying this philosophy to the attractive interaction, this paper will explore the quantum GLC.

2. Formalism

The GLC is regarded as a singularity appearing in the pressure-volume curve determined by the equation of state:

$$\frac{p}{k_BT} = \lim_{V\to\infty} \frac{\ln Z_V}{V}, \tag{1}$$

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$$\frac{\rho}{k_B T} = \lim_{V\to\infty} \frac{\partial}{\partial\mu} \left(\frac{\ln Z_V}{V}\right). \qquad (2)$$

This singularity comes from a singularity of $Z_V(\mu)$ in the right-hand side.

Let us write $Z_V(\mu)$ of the Boson system with an attractive contact interaction $g/V \sum_{p,p'} a_p^\dagger a_{-p}^\dagger a_{-p'} a_{p'}$, $(g < 0)$ in terms of the following perturbation expansion,

$$Z_V(\mu) = Z_0(\mu) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\beta_1 \cdots \int_0^\beta d\beta_n \times < TH_{it}(\beta_1)\cdots H_{it}(\beta_n) >_c . \qquad (3)$$

$Z_0(\mu)$ in the right-hand side represents the grand partition function of the free Bose gas. Using the symmetrized density matrix, $Z_0(N)$ has a following form in coordinate space,

$$Z_0(N) = \frac{1}{N!\lambda^{3N}} \int \sum_p \exp\left[-\frac{m}{2\beta\hbar^2} \sum_i (x_i - Px_i)^2\right] \times \rho(x_1\cdots x_N) dx_1\cdots dx_N, \qquad (4)$$

where P denotes the permutation, ρ a density distribution, and λ the thermal wavelength.

Feynman regarded $Z_0(\mu)$ as a random assembly of the many-body wave functions including s Bosons, in which an interchanging of s particles leaves the wave function unaltered, and he expressed $Z_0(\mu)$ as follows,

$$Z_0(\mu) = \exp\left(-\sum_s \frac{e^{\beta\mu s}}{s}\right) \prod_s \exp\left(-\frac{V}{\lambda^3}\frac{e^{\beta\mu s}}{s^{5/2}}\right). \qquad (5)$$

The first and second term in the right-hand side corresponds to the $p = 0$ and $p \neq 0$ Bose particles respectively. Using an identity $-\sum_{s=1}^{\infty}(-x)^s/s = \ln(1+x)$ in the first term, one obtains the well-known expression which shows the divergence of $Z_0(\mu)$ (BEC). In the normal phase, as the size of the many-body wave function increases, its contribution to $Z_0(\mu)$ decreases proportionally to $\exp(\beta\mu s)$ for the $p = 0$ Boson, and to $\exp(\beta\mu s)/s^{1.5}$ for $p \neq 0$ Boson. (Here we neglect a factor $1/s$ due to the rotational symmetry). At the BEC transition temperature T_0 ($\mu(T_0) = 0$), its size-dependence disappears for the $p = 0$ Boson, so that the macroscopic wave function contributes to $Z_0(\mu)$ as equally as the microscopic one.

The interaction is normally considered using the linked-cluster expansion. In the vicinity of the BEC transition point, however, the diagram which reflects the macroscopic many-body wave function plays an important role in $< TH_{it}(\beta_1)\cdots H_{it}(\beta_n) >_c$. Following the Bose statistics, the particle lines belonging to different bubble diagrams in it but having a same momentum p must be exchanged. Such a procedure gives us a polygon-like diagram which reflects the many-body wave function. At each vertex of the polygon, an interaction line appears. In $< TH_{it}(\beta_1)\cdots H_{it}(\beta_n) >_c$, these polygons being connected by the interaction lines form a cluster. A contribution of the polygon including $2s$ particles with the same p has a following form [2][3],

$$K_s = \frac{1}{V}\sum_{l,p}\left(-\frac{g}{\beta V}\frac{1}{(\epsilon_p-\mu)^2 + (\frac{\pi l}{\beta})^2}\right)^s. \qquad (6)$$

Summing various K_s's over all possible combinations in Eq.(3), one obtain a following $Z_V(\mu)$

$$Z_V(\mu) = Z_0(\mu)\frac{V}{2}\sum_{s=1}^{\infty} K_s \prod_s \exp\left(\frac{VK_s}{2s}\right). \qquad (7)$$

Since all possible ways of connecting K_s by the interaction lines in $< TH_{it}(\beta_1)\cdots H_{it}(\beta_n) >_c$ has an order of n for each distribution of K_s, the second term $V/2\sum_s K_s$ appears. We define x by $K_s = 1/V\sum_{l,p} x^s$. Using Eq.(7) in Eq.(2) and summing over s, one gets a following equation of states,

$$\frac{1}{v} = \frac{1}{\lambda^3} g_{3/2}(e^{\beta\mu}) + \frac{k_BT}{2V}\sum_{p,l}\frac{1}{1-x}\frac{\partial x}{\partial\mu} + \frac{k_BT}{V}\left(\sum_{p,l}\frac{1}{1-x}\right)^{-1}\sum_{p,l}\frac{1}{(1-x)^2}\frac{\partial x}{\partial\mu}, \qquad (8)$$

where $g_{3/2}(y) = \sum_n y^n/n^{3/2}$.

In cooling and compressing the system, the negative μ approaches zero. When such a condition is satisfied,

$$(\epsilon_p - \mu)^2 + (\frac{\pi l}{\beta})^2 + \frac{g}{\beta V} = 0, \qquad (9)$$

the integral over p in the second and the third term of the right-hand side of Eq.(8) diverges. Exactly, when μ and T satisfies a condition $\mu_c = -\sqrt{|g|k_BT_c/V}$, this divergence occurs first in the $l = 0$ term for $p = 0$ in Eq.(8). The value of μ_c and T_c is determined by this condition and the $\mu(T)$. In view of $\mu_c < 0$, $T_0 < T_c$ [4].

This divergence leads to a gas-liquid coexistence line characteristic of the GLC in the $P - V$ curve. Using this μ_c and T_c in $e^{\beta\mu s}$ of Eq.(5), one can estimate the critical size-distribution of the many-body wave function at the occurrence of the GLC as $\exp\left(-\sqrt{(|g|/V)/(k_BT_c)} \times s\right)$.

References

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