Correlation measures of the Calogero-Sutherland model at $T = 0$

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Abstract

We exploiting the exact results of the exact solution for the ground state of the one-dimensional spinless quantum gas of Fermions with μ/x_{ij}^2 particle-particle interaction, and analyze particle-number fluctuations for $\mu = -1/4,\ 0,$ and 2. These are suppressed by repulsive interaction ($\mu > 0$), enhanced by attraction ($\mu < 0$), and may therefore measure the kind and strength of correlation. Other recently proposed purely quantum-kinematical measures of the correlation strength arise from the small-separation behavior of the pair density or from the non-idempotency of the momentum distribution and its large-momenta behavior.

Key words: Theories and models of many-electron systems, Fermions in reduced dimensions, Fluctuation phenomena, Collective effects

In the following the fluctuation-correlation analysis of our previous papers [1] is summarized and extended. Usually the quantum-mechanical many-body problem can be solved only approximately. Therefore, rare exceptions of exactly solvable model systems are highly interesting for general discussions of fluctuations and correlations. The Calogero-Sutherland (CS) model

$$\hat{H} = \sum_i^N \frac{1}{2}\left(-\frac{\partial^2}{\partial x_i^2}\right) + \sum_{i<j}^N \frac{\nu^2 - \frac{1}{4}}{x_{ij}^2}, \quad x_{ij} = |x_i - x_j| \quad (1)$$

is such a system [2] describing interacting spinless fermions in one dimension. In the thermodynamic limit, the CS ground state has only two parameters, the interaction strength parameter $\nu > 0$ (attraction for $\nu < 1/2$, repulsion for $\nu > 1/2$) and the particle density n or the Fermi wave number $k_{\rm F} = \pi n$. Its quantum-kinematical characteristics are (i) the pair density (PD) $g(x_{12})$ with the interparticle distance x_{12} measured in units of $k_{\rm F}^{-1}$ and (ii) the momentum distribution (MD) $n(k)$ with the momentum k measured in units of $k_{\rm F}$. Important properties (sum rules) of the PD and the MD are $g(0) = 0$, $g(\infty) = 1$, $\int \frac{dx}{\pi}\,[1 - g(x)] = 1$, $0 \le n(k) \le 1$ (non-idempotency) and $\int_0^\infty dk\, n(k) = 1$. $\nu = 1/2$ describes the ideal spinless one-dimensional Fermi gas with the ideal Fermi hole $g(x) = 1 - (\sin x/x)^2$ and the ideal Fermi ice block $n(k) = \Theta(1 - k)$.

Pair Density, Fluctuations, and Correlations: The PD is known analytically [2] for $\nu = 0$ (attraction)

$$g(x) = 1 - \left(\frac{\sin x}{x}\right)^2 + \mathrm{Si}(x)\frac{d}{dx}\frac{\sin x}{x} - \frac{\pi}{2}\frac{d}{dx}\frac{\sin x}{x} \quad (2)$$

and for $\nu = 3/2$ (repulsion)

$$g(x) = 1 - \left(\frac{\sin 2x}{2x}\right)^2 + \mathrm{Si}(2x)\frac{d}{d(2x)}\frac{\sin 2x}{2x}. \quad (3)$$

The exponent of the small-x behavior $g(x \to 0) \sim x^{2+\alpha}$ is $\alpha = 2\nu - 1$. The deviations of (2) and (3) from the ideal Fermi hole indicate the attraction and repulsion induced correlation, respectively. Besides one may ask to what extent correlation influences particle-number fluctuations ΔN_X in a domain X, i.e., a certain interval of the x axis, where in the average there are $N_X = nX$ particles. These fluctuations are measured quantitatively by [3]

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$$\frac{(\Delta N_X)^2}{N_X} = 1 - \frac{1}{X}\int_0^X dx_1 \int_0^X dx_2 \frac{1-g(x_{12})}{\pi}. \tag{4}$$

The results are shown in Fig. 1, where also the case $\nu \to \infty$ ("strict" or "perfect" correlation) [3] is displayed, the oscillations of which have a weak precursor already for $\nu = 3/2$. The particle-number fluctuations are suppressed due to repulsive particle interaction [3], but enhanced due to attractive particle interaction. Correlation makes the particle-number distribution $P_X(N)$ with $N_X = \sum_N P_X(N)N$ and $(N^2)_X = \sum_N P_X(N)N^2$ more narrow for repulsion ($\nu > 1/2$) and more broad for attraction ($\nu < 1/2$). We remark that fluctuation enhancement (induced by attractive interaction) generally may support/cause clusterings (e.g., paramagnons prior the paramagnetic-to-ferromagentic phase transition). In our case, this tendency shows up in the sudden "fall-into-the-origin" at $\nu = 0$, where also the kinetic and the potential energy diverge (because of the large-k behavior of $n(k)$ and the small-x behavior of $g(x)$), but such that their sum is finite.

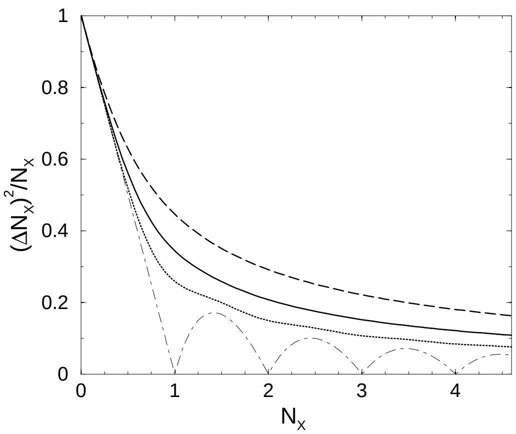

Fig. 1. Particle-number fluctuation $(\Delta N_X)^2/N_X$ in domains X of the CS model after Eq. (4) for $\nu = 0$ (dashed), 1/2 (solid), and 3/2 (dotted). The dashed-dotted line corresponds to $(\Delta N_X)^2/N_X$ for strict correlation. [3]

Momentum Distribution and Correlations: The MD for $\nu = 0$ and $\nu = 3/2$ is evaluated numerically by using the connection with random matrix theory as outlined in Refs. [2,1]. For $\nu \neq 1/2$ the interaction induced melting of the Fermi ice block near $k = 1$ is described by $|n(k) - 1/2| \sim |k-1|^\beta$ with a critical exponent β, which has been computed from conformal field theory to be $\beta = \frac{1}{4}(1-2\nu)^2/(1+2\nu)$ [2]. Other from $n(k)$ derived correlation measures are the (2nd-order) non-idempotency $c = \int_0^\infty dk\; n(k)[1-n(k)]$ [4,5], the correlation entropy $s = \int_0^\infty dk\; (-1)\{n(k)\ln n(k) + [1-n(k)]\ln[1-n(k)]\}$ [5,6], and the tail normalization $n_{\rm tail} = \int_1^\infty dk\; n(k)$. Note the invariance of c, s, and of the critical behavior near the Fermi edge under the exchange $n(k) \leftrightarrow 1-n(k)$. This particle-hole symmetry is important for the recently developing density matrix functional theory [7,8]. $n(k)$ and $1-n(k)$ are the probabilities for the momentum state k to be occupied and empty, respectively. The entropy of this probability 'distribution' is just the integrand of s which is the sum of all these entropies. [6]. Note that the above introduced PD exponent α also describes the MD asymptotics $n(k \to \infty) \sim 1/k^{4+\alpha}$. Fig. 2 shows the interaction induced MD characteristics $\alpha, \beta, c, s, n_{\rm tail}$ vs. ν. Only the exponent α is sensitive against the sign of the particle interaction (attraction, repulsion) like the particle-number fluctuations $(\Delta N_X)^2$, cf. Fig. 1.

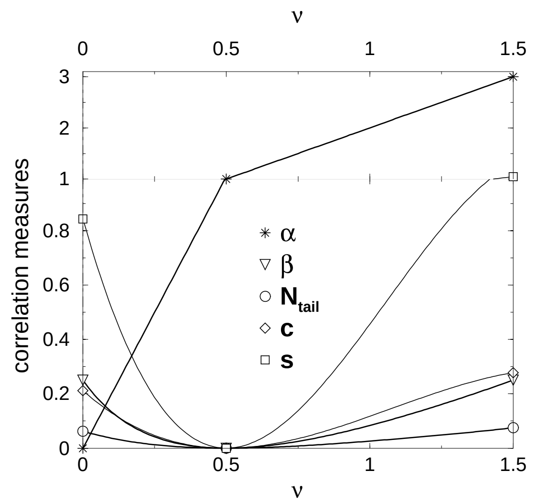

Fig. 2. Correlation measures shown as functions of ν. The solid lines are guides to the eye only. The thin dashed-dotted line indicates the "fall-into-the-origin" at $\nu = 0$.

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