Universality of the ground states of rotating bosonic atoms in magnetic traps

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Abstract

Trapped bosonic atoms with manipulated interaction strength near the sign flip of the scattering length are considered. Analytic solution for the rotating ground states is obtained. The form of the ground states at any angular momentum is universal, and it do not depend on the details of the interaction. The ground states are either *collective rotations* or *condensed vortex states*, depending on the sign of the *modified Born scattering length*.

Key words: Bose-Einstein condensation, trapped atoms, arbitrary interaction

Recently[1], it has become possible to manipulate the effective interaction between trapped Bose atoms[2]. These techniques allow study the weakly interacting Bose condensates. Of special interest are the ground states of such systems at a fixed value of the angular momenta L, the "yrast states" [3,4].

The Hamiltonian of N spinless bosons in a two-dimensional symmetric harmonic trap reads

$$H = \sum_{i}^{N} \left(\frac{\mathbf{p_i^2}}{2m} + \frac{m\omega^2 \mathbf{r_i^2}}{2} \right) + \sum_{i>j}^{N} U(r_{ij}^2/2) \quad (1)$$

The first term describes noninteracting particles. U is the two-body interaction, $r_{ij}=|\mathbf{r}_i-\mathbf{r}_j|$. In the limit $\hbar\omega \gg V$, the ground state $|0_L)$ as a function of $L=|L_{xy}|$ can be obtained from the diagonalization of the interaction within the space of degenerate states[3,4]

$$S z_1^{l_1} z_2^{l_2} ... z_N^{l_N} e^{-\sum \frac{z_k \bar{z}_k}{2}}, \quad l_1 + l_2 + ... + l_N = L, \quad (2)$$

where $z_i(\bar{z}_i)=x_i \pm i y_i$, S is the symmetrization operator. Hereafter, we set $\hbar=m=\omega=1$. The dimensionality of the basis grows exponentially with L[3,4].

The problem was usually studied for the contact interaction $\pm\delta(\mathbf{r})$[3,4]. Universality classes of interactions of more general form have been studied in [5][6],[7]. Here, we solve the problem for *arbitrary interaction* U_{phys} satisfying reasonable physical requirements: We assume that the force $F \equiv \frac{-dU}{dt}$ is positive at small separation and change sign at some distance smaller than the trapping size $t=1$.

Within the subspace (2), the Hamiltonian (1) is

$$\hat{H} = N + L + \hat{V}, \quad \hat{V} = N_p \bar{u} + N_p S w(\hat{l}_{12}) S, \quad (3)$$

Here, $N_p = \frac{N^2-N}{2}$ is the number of interacting pairs, $\bar{u}$ is the constant $\int_0^\infty U(t) e^{-t} dt$. The operator term is the work by the "force" $\phi_l F$ to separate the two particles, $w(l) = \int_0^\infty \phi_l F(t) dt$. Here, $\phi_l \equiv \frac{\gamma(l,t)}{\Gamma(l)}$ with γ the incomplete gamma function; $\hat{l}_{ij} \equiv \frac{1}{2}(a_i^+ - a_j^+)\ (a_i - a_j)$ is the interparticle angular momentum in terms of the ladder operators $a_i^+ = \frac{z_i}{2} - \frac{\partial}{\partial z_i^*}$ and $a_i \equiv (a^+)^\dagger$.

The total *internal angular momentum* $J=\sum_{i>j} \hat{l}_{ij}$ commutes with both $L \equiv \sum_i a_i^+ a_i$ and $\hat{H}$. Its eigenvalues are $J=\frac{Nj}{2}$, j=0,2,3,..,L[5].

We now split the interaction $\hat{V}$ into two parts $\hat{V} = \hat{V}_0 + \hat{V}_S$ [5] such that (i) the first term is simple enough to find its lowest eigenvalue E_0 and its associated eigen-

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state $|0)$, $\hat{V}_0|0)=E_0|0)$. The state $|0)$ will also be the ground state of the total interaction $\hat{V}=\hat{V}_0+\hat{V}_S$ if (ii) $\hat{V}_S$ is *non-negative definite*, $\hat{V}_S\geq 0$, and (iii) $|0)$ is annihilated by V_S[5] $\hat{V}_S|0)=0$. The operator V_0 can be sought in the form

$$\hat{V}_0 = N_p S v_0(\hat{l}_{ij})S, \quad v_0(l) = c_0 + c_1 l + c_2 l^2 + ..., \quad (4)$$

with c_n unknown numbers. Any operator $v(\hat{l}_{12})$ is diagonal in the states $|\mu\rangle=(\frac{z_1-z_2}{\sqrt{2}})^{l_1}(\frac{z_1+z_2}{\sqrt{2}})^{l_2}\ z_3^{l_3}...z_N^{l_N}|0\rangle$ with eigenvalues $v(l)$, l=0,1,2,...,L. Odd-l eigenvectors are annihilated by S (Cf. Eq.(2)). With appropriate choice of c_m, all even (l=2n) eigenvalues of v_S, $\lambda_n=w(2n)-v_0(2n)$ can be made non-negative. This will guarantee $V_S\geq 0$[5]. The candidates for the state $|0)$ satisfying (i) and (iii) can be sought among the linear combinations of those $S|\mu\rangle$'s which obey $v_S|\mu\rangle=0$. At $L \leq N$, the solution for c_m can be found in the form $c_0=\bar{u}$, $c_1=-\frac{1}{2}\Delta_2\theta(\Delta_2)$, $c_{m\geq 2}=0$, where $\theta(x)=\{0(x\leq 0),1(x>0)\}$ is the step function and $\Delta_{2n}\equiv -w(2n)$. The control eigenvalues are

$$\lambda_n = n\theta(\Delta_2)\Delta_2 - \Delta_{2n} \geq 0, \quad (5)$$

The inequalities hold for all interactions U of physical interest. The ground state and its energy are

$$|0_L) = e^{-\sum\frac{|z_i|^2}{2}}\hat{S}\prod_{j=1}^{L}\sum_{i=1}^{N}(z_i - z_j\theta(\Delta_2)), \quad (6)$$

$$E_0(L) = N + L + N_p\bar{u} - (NL/4)\Delta_2\theta(\Delta_2).$$

For the potentials with $\Delta_2<0$ and $\Delta_2<0$, the ground state (6) reduces to $|0_L^-)=e^{-\sum\frac{|z_i|^2}{2}}Z^L$ and $|0_L^+)=e^{-\sum\frac{|z_i|^2}{2}}\hat{S}\bar{z}_1\bar{z}_2...\bar{z}_L$, respectively. The two differ by the value of *discrete order parameter* J: we have $(0_L^-|J|0_L^-)=0$ and $(0_L^+|J|0_L^+)=\frac{NL}{2}$. At $L\gg 1$, the *condensed vortex state* $|0_L^+)$ reveals the Bose condensate[8], while the $|0_L^-)$, the "collective rotation state", does not[3]. The sign of the *modified Born scattering length*

$$\Delta_2 = \int_0^\infty dte^{-t}\left(1 - t^2/2\right)U(t) \quad (7)$$

separates the two universality classes of interaction. The transition between the two classes is possible when the form of the interaction U is continuously changed. At the critical point $\Delta_2=0$, the derivative $\partial E_0(L)/\partial\Delta_2$ has a jump $\frac{-NL}{4}$. Curiously, the behavior of $w(2n)$ near $\Delta_2=0$ resembles that of thermodynamic potential in a second-order phase transition [see Fig. 1].

We illustrate these features by using the two-parametric family of potentials $U_p=\frac{e^{-r^2/R^2}}{R^2}(1+a-\frac{2ar^2}{R^2})$. We obtain $w(l)= g_0-g_l$ where $g_l=(\zeta^{l+1}/R^2)$ $[1+a$ $(1-4(l+1)\ \zeta)/R^2]$ with $\zeta=R^2/(2+R^2)$. In Fig. 1, we show the *phase diagram* for U_p in the $a-R$ plane. The separatrix is defined by equiation $\Delta_2=0$ which gives

$$a = \frac{(1+R^2)(2+R^2)}{2+R^2-3R^4}. \quad (8)$$

It separates the two classes of potentials with qualitatively different ground states $|0_L^-)$ and $|0_L^+)$.

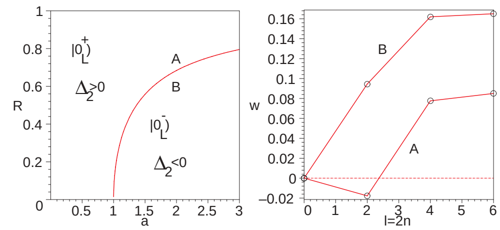

Fig. 1. Left: Phase diagram for family of the potentials U_p on the $a-R$-plane. The curve is the separatrix Δ_2=0 (8). Right: The quantity $w(2n)$ as a function of $2n$ (symbols connected by a line) for the two sets of parameters A and B, marked by their positions on the left panel.

The control eigenvalues (5) are seen non-negative throughout. For values of a smaller than $a(R)$ in (8), $\Delta_2\geq 0$ and $\lambda_n=n\Delta_2-\Delta_{2n}\geq 0$ is non-negative; for $a\geq a(R)$, $\Delta_2\leq 0$ and $\lambda_n=-\Delta_{2n}$ again is non-negative. Thus *all* the potentials U_p fall into two distinct classes depending on whether Δ_2 is positive or negative. The ground state is either collective rotation or a condensed vortex state. We obtained similar results for other two-parametric potential families in arbitrary spatial dimensionality[8].

This situation is in fact generic: the inequalities (5) are satisfied[8] within the class of interactions U_{phys}. The ground state is covered by the universal result (6).

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