Non-universal exponents at the Anderson transition

Hideaki Obuse [a], Kousuke Yakubo [a,1]
[a] *Department of Applied Physics, Hokkaido University, Sapporo 060-8628, Japan*

Abstract

Fluctuations of the correlation dimension D_2 describing multifractal properties of critical wavefunctions at the two- and three-dimensional Anderson transition points are studied by employing the forced-oscillator method and the box-counting procedure. We show that the width of the distribution function of D_2 over disorder realizations remains finite even in the thermodynamic limit. Similar results are obtained for the exponent α characterizing quantum diffusion at criticality. These imply that exponents defined at the critical point may not be universal. It is also shown that the scaling relation $D_2 = d\alpha$ (d is the spatial dimension) does not hold for individual samples, but is the case in a statistical sense.

Key words: Anderson transition; universality; multifractal

The Anderson metal-insulator transition is a quantum phase transition at the absolute zero temperature. Critical behaviors of this transition are characterized by several critical exponents as in the case of thermal critical phenomena. It is widely expected that these exponents do not depend on microscopic details of systems. This remarkable feature provides the universal nature of critical phenomena. The correlation dimension D_2 describing multifractality of critical wavefunctions at the Anderson transition point is believed to be one of such exponents. The exponent D_2 is, however, different from the localization-length exponent ν or the conductivity exponent s in the following sense. The exponent D_2 is defined *at* the critical point, while ν and s characterize the system *near* the critical point. This is a quite important difference for the universality of exponents.

The exponent D_2 is defined by the correlation function of the critical wavefunction $\psi(\boldsymbol{r})$ as

$$S(\boldsymbol{r}_1, \boldsymbol{r}_2) \equiv \left\langle |\psi(\boldsymbol{r}_1)|^2 \, |\psi(\boldsymbol{r}_2)|^2 \right\rangle_{\rm sp} \propto |\boldsymbol{r}_1 - \boldsymbol{r}_2|^{-(d-D_2)}, \qquad (1)$$

where d is the spatial dimension and $\langle \cdots \rangle_{\rm sp}$ denotes the spatial average. If the spatial average is equivalent to the ensemble average $\langle \cdots \rangle_{\rm en}$ over disorder realizations, the exponent D_2 can be also expressed by

$$P_2(L) = \left\langle \int |\psi_L(\boldsymbol{r})|^4 \, d\boldsymbol{r} \right\rangle_{\rm en} \propto L^{-D_2}, \qquad (2)$$

where P_2 is the inverse participation ratio and ψ_L represents the critical wavefunction in a system of size L. The correlation dimension D_2 obtained by Eq. (2) does not fluctuate over samples, because $P_2(L)$ is a statistical quantity of the disorder ensemble. The equivalence between $\langle \cdots \rangle_{\rm sp}$ and $\langle \cdots \rangle_{\rm en}$ is, however, not obvious because of the infinite correlation length at criticality. There exists a possibility that D_2 defined by Eq. (1) fluctuates over samples even in the thermodynamic limit, which means that D_2 is not a universal exponent. Similar arguments hold also for any exponents defined at the critical point.

The aim of this paper is clarifying whether exponents defined at the Anderson transition point depend on disorder realizations. For this purpose, we calculate numerically the correlation dimension D_2 of critical wavefunctions and the exponent α describing quantum diffusion at criticality for the two- and three-dimensional Anderson transitions.

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[1] E-mail:yakubo@eng.hokudai.ac.jp

Two classes of systems are studied here. One is the two-dimensional symplectic class (2DSC) in which a system has the time-reversal symmetry and no spin-rotational symmetry. Electrons in a system belonging to this class are described by the tight-binding Hamiltonian with the critical disorder $W_c = 5.86V$, where V is the hopping energy[1,2]. Another is the three-dimensional orthogonal class (3DOC). A system belonging to this class has the time-reversal and spin-rotational symmetries. The critical disorder W_c is chosen to be $16.5V$[3]. We have calculated critical wavefunctions for a number of disorder realizations by using the forced-oscillator method (FOM)[4]. This method enables us to calculate eigenvectors of large matrices with saving computational resources. The correlation dimension D_2 for individual critical wavefunction is obtained by the box-counting method. In the case of the 2DSC, the system size ranges from 12 to 120, and the number of disorder realizations is of the order of 10^3. For the 3DOC, the system size varies from 8 to 30, and the number of samples for each size is of the order of 10^4. We set the periodic boundary condition for both systems. In quantum diffusion calculations, we use 100 samples of size $L = 1,001$ for the 2DSC and $3,600$ samples of size $L = 101$ for the 3DOC. The Chebyshev expansion technique[5] is employed to compute the time evolution of a wave packet.

Figure 1 shows the system size dependence of standard deviations σ_{D_2} of the distribution function of D_2. Since the correlation length diverges at the critical point, σ_{D_2} depends on L only via $L/\xi_{\rm irr}$, where $\xi_{\rm irr}$ is an irrelevant length. Assuming that the irrelevant length $\xi_{\rm irr}$ is related to the irrelevant scaling field $\zeta_{\rm irr}$ as $\xi_{\rm irr} \propto \zeta_{\rm irr}^{1/y}$, the standard deviation σ_{D_2} can be written as $\sigma_{D_2} = f[(L/\xi_{\rm irr})^{-y}]$, where f is a scaling function and y is a positive exponent. For $L \gg \xi_{\rm irr}$, the scaling function can be expanded as

$$\sigma_{D_2}(L) = \sigma^*_{D_2} + AL^{-y} \ . \qquad (3)$$

The vertical axis of Fig. 1 is $\sigma_{D_2}(L) - \sigma^*_{D_2}$ and the slope of data represents the value of the exponent y. From our results shown in Fig. 1, $\sigma^*_{D_2} = 0.041 \pm 0.017$ and $y = 0.48 \pm 0.15$ for the 2DSC and $\sigma^*_{D_2} = 0.142 \pm 0.051$ and $y = 0.43 \pm 0.16$ for the 3DOC. The quantity $\sigma^*_{D_2}$ is the width of the distribution function of D_2 in the limit of $L \to \infty$. The fact that $\sigma^*_{D_2}$ is finite implies that values of D_2 fluctuate over disorder realizations even in the thermodynamic limit. The inset of Fig. 1 shows similar results for the exponent α. In this case, the scaling should be considered for the diffusion time t instead of the system size L. We obtain $\sigma^*_\alpha = 0.045 \pm 0.001$ and $y = 0.44 \pm 0.01$ for the 2DSC and $\sigma^*_\alpha = 0.085 \pm 0.001$ and $y = 0.33 \pm 0.01$ for the 3DOC. From these results, we suggest that any exponents defined at the Anderson transition point have finite fluctuations.

It is known that D_2 is related to α by the relation

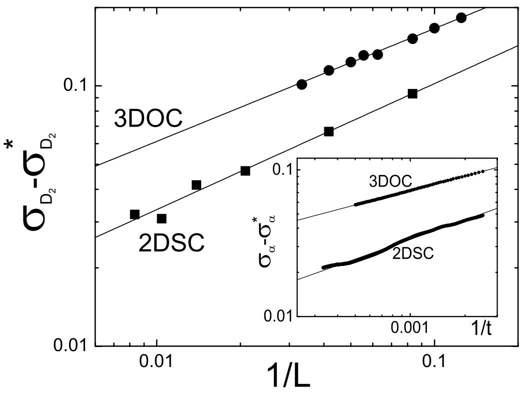

Fig. 1. System size dependence of the standard deviation σ_{D_2} of the correlation dimension D_2. The inset shows the diffusion time dependence of the standard deviation σ_α of the exponent α for quantum diffusion.

$$D_2 = d\alpha. \qquad (4)$$

If Eq. (4) holds for individual samples, we should have $\sigma^*_{D_2} = d\sigma^*_\alpha$. Values of standard deviations calculated here do not satisfy this relation. In fact, plotting calculated D_2 and α for each sample by a point in the D_2-α plane, data points scatter around the straight line $D_2 = d\alpha$. The typical values of the distribution functions of D_2 and α are, however, $D_2^{\rm typ} = 1.64$ and $\alpha^{\rm typ} = 0.83$ for the 2DSC and $D_2^{\rm typ} = 1.36$ and $\alpha^{\rm typ} = 0.44$ for the 3DOC, which satisfy Eq. (4). Here the typical value is defined as the geometric mean of the distributing exponents. These results indicate that Eq. (4) holds in a statistical sense, but not for individual realization of disorder.

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