Flow equation renormalization of a spin-boson model with a structured bath

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Abstract

We discuss the dynamics of a spin coupled to a damped harmonic oscillator. This system can be mapped to a spin-boson model with a structured bath, i.e. the spectral function of the bath has a resonance peak. We diagonalize the model by means of infinitesimal unitary transformations (*flow equations*), thereby decoupling the small quantum system from its environment, and calculate spin-spin correlation functions.

Key words: flow equations; quantum dissipative systems, spin-boson, structured bath

1. Introduction - Model

Recently a new strategy for performing measurements on solid state (Josephson) qubits was proposed which uses the entanglement of the qubit with states of a damped oscillator [1], with this oscillator representing the plasma resonance of the Josephson junction. This system of a spin coupled to a damped harmonic oscillator (see Fig. 1) can be mapped to a standard model for dissipative quantum systems, namely the spin-boson model [2]. Here the spectral function governing the dynamics of the spin has a resonance peak. Such structured baths were also discussed in connection with electron transfer processes [2]. We use the flow equation method introduced by Wegner [3] to analyze the system shown in Fig. 1, consisting of a two-level system coupled to a harmonic oscillator Ω, which is coupled to a bath of harmonic oscillators:

$$\tilde{\mathcal{H}} = -\frac{\Delta_0}{2}\sigma_x + \Omega B^\dagger B + g(B^\dagger + B)\sigma_z + \sum_k \tilde{\omega}_k \tilde{b}_k^\dagger \tilde{b}_k + (B^\dagger + B)\sum_k \kappa_k(\tilde{b}_k^\dagger + \tilde{b}_k) + (B^\dagger + B)^2 \sum_k \frac{\kappa_k^2}{\tilde{\omega}_k},$$

with the spectral function $J(\omega) \equiv \sum_k \kappa_k^2 \delta(\omega - \tilde{\omega}_k) = \Gamma\omega$. This system can be mapped to a spin-boson model [2]

$$\mathcal{H} = -\frac{\Delta_0}{2}\sigma_x + \frac{1}{2}\sigma_z \sum_k \lambda_k(b_k^\dagger + b_k) + \sum_k \omega_k b_k^\dagger b_k, \quad (1)$$

where the dynamics of the spin depends only on the spectral function $J(\omega) \equiv \sum_k \lambda_k^2 \delta(\omega - \omega_k)$ given by

$$J(\omega) = \frac{2\alpha\omega\Omega^4}{(\Omega^2 - \omega^2)^2 + (2\pi\Gamma\omega\Omega)^2} \text{ with } \alpha = \frac{8\Gamma g^2}{\Omega^2}. \quad (2)$$

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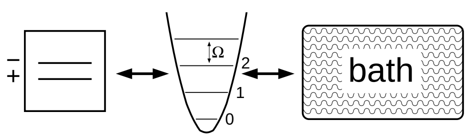

Fig. 1. A two-level system is coupled to a damped harmonic oscillator with frequency Ω.

2. Method - Results

Using the flow equation technique we approximately diagonalize the Hamiltonian $\mathcal{H}$ [Eq.(1)] by means of infinitesimal unitary transformations. The continuous sequence of unitary transformations $U(l)$ is labelled by a flow parameter l. Applying such a transformation to a given Hamiltonian, this Hamiltonian becomes a function of l: $\mathcal{H}(l) = U(l)\mathcal{H}U^{\dagger}(l)$. Here $\mathcal{H}(l=0) = \mathcal{H}$ is the initial Hamiltonian and $\mathcal{H}(l=\infty)$ is the final diagonal Hamiltonian. Usually it is more convenient to work with a differential formulation

$$\frac{d\mathcal{H}(l)}{dl} = [\eta(l), \mathcal{H}(l)] \quad \text{with} \quad \eta(l) = \frac{dU(l)}{dl}U^{-1}(l). \quad (3)$$

Using the flow equation approach one can decouple system and bath by diagonalizing $\mathcal{H}(l=0)$ [4]:

$$\mathcal{H}(l=\infty) = -\frac{\Delta_\infty}{2}\sigma_{\mathrm{x}} + \sum_{\mathrm{k}} \omega_{\mathrm{k}} b_{\mathrm{k}}^{\dagger} b_{\mathrm{k}}. \quad (4)$$

Here Δ_∞ is the renormalized tunneling frequency. For the generator of the flow we choose the Ansatz [4]:

$$\eta = \sum_{\mathrm{k}} \left(\mathrm{i}\sigma_{\mathrm{y}}\Delta(b_{\mathrm{k}} + b_{\mathrm{k}}^{\dagger}) + \sigma_{\mathrm{z}}\omega_{\mathrm{k}}(b_{\mathrm{k}} - b_{\mathrm{k}}^{\dagger})\right) \frac{\lambda_{\mathrm{k}}}{2} \left(\frac{\Delta - \omega_{\mathrm{k}}}{\Delta + \omega_{\mathrm{k}}}\right) + \frac{\Delta}{2} \sum_{\mathrm{q,k}} \lambda_{\mathrm{k}}\lambda_{\mathrm{q}} I(\omega_{\mathrm{k}}, \omega_{\mathrm{q}}, l)(b_{\mathrm{k}} + b_{\mathrm{k}}^{\dagger})(b_{\mathrm{q}} - b_{\mathrm{q}}^{\dagger}), \quad (5)$$

$$\text{with} \quad I(\omega_{\mathrm{k}}, \omega_{\mathrm{q}}, l) = \frac{\omega_{\mathrm{q}}}{\omega_{\mathrm{k}}^2 - \omega_{\mathrm{q}}^2} \left(\frac{\omega_{\mathrm{k}} - \Delta}{\omega_{\mathrm{k}} + \Delta} + \frac{\omega_{\mathrm{q}} - \Delta}{\omega_{\mathrm{q}} + \Delta}\right).$$

The flow equations for the effective Hamiltonian [Eq. (4)] then take the following form:

$$\frac{\partial J(\omega, l)}{\partial l} = -2(\omega - \Delta)^2 J(\omega, l) + 2\Delta J(\omega, l) \int \mathrm{d}\omega' J(\omega', l) I(\omega, \omega', l), \quad (6)$$

$$\frac{d\Delta}{dl} = -\Delta \int \mathrm{d}\omega J(\omega, l) \frac{\omega - \Delta}{\omega + \Delta}. \quad (7)$$

The unitary flow diagonalizing the Hamiltonian generates a flow for $\sigma_{\mathrm{z}}(l)$ which takes the structure

$$\sigma_{\mathrm{z}}(l) = h(l)\sigma_{\mathrm{z}} + \sigma_{\mathrm{x}} \sum_{\mathrm{k}} \chi_{\mathrm{k}}(l)(b_{\mathrm{k}} + b_{\mathrm{k}}^{\dagger}), \quad (8)$$

where $h(l)$ and $\chi_{\mathrm{k}}(l)$ obey the differential equations

$$\frac{dh}{dl} = -\Delta \sum_{\mathrm{k}} \lambda_{\mathrm{k}}\chi_{\mathrm{k}} \frac{\omega_{\mathrm{k}} - \Delta}{\omega_{\mathrm{k}} + \Delta}, \quad (9)$$

$$\frac{d\chi_{\mathrm{k}}}{dl} = \Delta h \lambda_{\mathrm{k}} \frac{\omega_{\mathrm{k}} - \Delta}{\omega_{\mathrm{k}} + \Delta} + \sum_{\mathrm{q}} \chi_{\mathrm{q}}\lambda_{\mathrm{k}}\lambda_{\mathrm{q}}\Delta I(\omega_{\mathrm{k}}, \omega_{\mathrm{q}}, l). \quad (10)$$

One can show that the function $h(l)$ decays to zero as $l \to \infty$. Therefore the observable σ_{z} decays completely into bath operators [4].

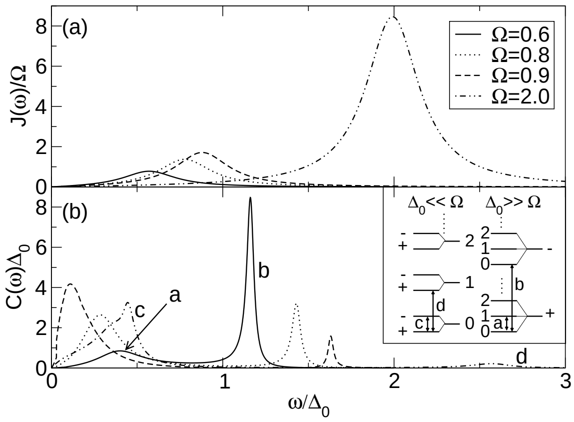

Fig. 2. (a) Different effective spectral functions $J(\omega, l=0)$ and (b) the corresponding $C(\omega)$ for $\Delta\Gamma = 0.06$ and $\alpha = 0.15$. The inset shows the term scheme of a two-level system coupled to a harmonic oscillator for the two limits $\Delta_0 \ll \Omega$ and $\Delta_0 \gg \Omega$.

We integrated the flow equations numerically in order to calculate the Fourier transform, $C(\omega)$, of the spin-spin correlation function

$$C(t) \equiv \frac{1}{2}\langle\sigma_{\mathrm{z}}(t)\sigma_{\mathrm{z}}(0) + \sigma_{\mathrm{z}}(0)\sigma_{\mathrm{z}}(t)\rangle. \quad (11)$$

$C(t)$ can be used to calculate dephasing and relaxation times for measurements on qubits [1]. Fig. 2(a) shows $J(\omega, l=0)$ and Fig. 2(b) $C(\omega)$ for different values of Ω. $C(\omega)$ displays a double-peak structure, which can be understood from the term scheme shown in the inset. The arrows indicate the transitions responsible for the peaks in $C(\omega)$. Additional structure of $C(\omega)$ due to higher order transitions in the term scheme is not seen in Fig. 2. This is due to our Ansatz for $\sigma_{\mathrm{z}}(l)$ [see Eq.(8)], which does not include the corresponding higher order terms. However, we do not expect the additional peaks to have much weight, as the sum rule [4] for the total spectral weight is fulfilled with an error of less than 5% for all the plots in Fig. 2(b). We leave the extension of the Ansatz for $\sigma_{\mathrm{z}}(l)$ for future work.

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