Equation of state from hydrodynamic modes in dense trapped ultracold gases

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Abstract

Hydrodynamic modes for ultracold gases at any general density are considered. By writing the equations for the mode frequencies in a convenient way, we show that it is possible to determine the equation of state of the dense gas from the knowledge of the hydrodynamic frequencies. As an example, we investigate the case of two equal fermionic populations in different hyperfine states with attractive interactions.

Key words: Fermion gases; Bose-Einstein condensation; hydrodynamic modes

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1. Introduction

We consider here the possibility of ultracold gases in dense situations, where interatomic distance and scattering length are comparable, and study the mode frequencies in the hydrodynamic regime. Indeed, most of the recent work on ultracold gases considered dilute situations. Dense systems are interesting e.g. in the search of a BCS transition, since the critical temperature will be higher in this regime [1,2]. We show here that the knowledge of the mode frequencies leads directly to the determination of the equation of state of the gas, even in non dilute situations.

2. Theory and models

The basic starting equations are the Euler equation $m\,d\mathbf{v}/dt = -\nabla(\mu(n) + V)$ and the particle conservation equation $\partial n/\partial t + \nabla \cdot (n\mathbf{v}) = 0$. We consider here to be specific an isotropic harmonic potential $V(r) = \frac{1}{2}\,m\,\Omega^2 r^2$, even though our approach could be generalized to anisotropic situations. The only physical ingredient (apart from the trap frequency Ω) is therefore included in the chemical potential density dependance $\mu(n)$. The equilibrium particle density $n_0(r)$ satisfies $\mu(n_0(r)) + V(r) = \tilde{\mu}$, where $\tilde{\mu}$ is the overall chemical potential. Linearizing these equations around equilibrium, one finds a linear second order differential equation for the density fluctuation $n_1(\mathbf{r})e^{-i\omega\,t} = n(\mathbf{r},t) - n_0(r)$ oscillating at frequency ω. It is actually convenient to make the change $n_1(\mathbf{r}) = (\partial\mu/\partial n_0)^{-1} Y_{lm}(\theta,\varphi)\,r^l v(r)$, so that one gets the following general equation for the hydrodynamic modes in an isotropic harmonic trap:

$$rv'' + \Big[2(l+1) + r\,L'(r)\Big]\,v' - (\nu^2 - l)L'(r)v = 0 \quad (1)$$

where we have set $\nu^2 = \omega^2/\Omega^2$ and $L(r) = \log(n_0(r))$. One can then easily check on this equation that, whatever the equation of state $\mu(n)$, there are modes at frequency $\omega = \sqrt{l}\,\Omega$ (corresponding to $\nu^2 = l$ and $v = 1$). They correspond physically to the dipolar mode ($l = 1$) or to surface modes ($l \geq 2$) [3]. First Eq.(1) is unchanged if we rescale r by the radius of the cloud, so that we can consider $0 < r < 1$. We can then notice that Eq.(1) is only slightly modified by the change of variable $y = r^\alpha$, if one makes the same change for $L(r)$. The new equation for the modes becomes:

$$y\frac{d^2v}{dy^2} + (1 + \frac{2\,l+1}{\alpha} + y\frac{d\,L}{dy})\frac{dv}{dy} - \frac{\nu^2 - l}{\alpha}\frac{dL}{dy}v = 0 \quad (2)$$

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Then we consider [6] the two parameters family of function $L(y) = -p/(1-y)$ corresponding to a chemical potential of the form $\mu(n) \propto 1-(1-(n/n(0))^{1/p})^{2/\alpha}$, where $n(0)$ is the density at the center $r = 0$ and α, p are two parameters. In this case, Eq.(2) turns out to be the hypergeometric differential equation. The eigenmodes are

$$\frac{\omega^2}{\Omega^2} = l + \frac{\alpha}{p}\, n\, (n + p + \frac{2\,l+1}{\alpha}) \qquad (3)$$

with $n = 1, 2, \cdots$. The regular solutions are polynomials in the variable y. Eq.(3) agrees with the known results for bosons [4] ($\alpha = 2$ and $p = 1$) and free fermions [5] ($\alpha = 2$ and $p = 3/2$).

This two parameters family can in fact be extended, by considering models with $L(y) = -\frac{1}{1-y}(\sum_{k=0}^{K} p_k y^k)$. The series expansion of $v(y)$ is very rapidly convergent [6]. In practice, one can truncate the series expansion in order to get the modes and their frequencies. These 'quasipolynomial solutions' therefore enable very easy numerical calculations of the modes.

These two family of models can give the mode frequencies by fitting a given function $\mu(n)$ by a model chemical potential. Conversely, the knowledge of the mode frequencies can determine the models parameters (α, p_0, $p_1, \cdots$) and lead to an approximate equation of state $\mu(n)$.

3. A specific example

As an example, we consider the specific case of two equal populations of fermions in different internal hyperfine states. We assume an attractive interaction between atoms in two different internal states with an interaction g, related to the negative diffusion length a by $g = 4\pi\hbar^2\, a/m$. We consider the Hartree approximation, where the chemical potential is given by $\mu(n) = \hbar^2 k_f^2/2m - |g|n/2$ with $n = \frac{1}{3\pi^2} k_f^3$. The dimensionless parameter $\lambda = \frac{2}{\pi} k_f(0)|a|$, where $k_f(0)$ is the equilibrium Fermi wavevector at the center $r = 0$, goes from 0 for the very dilute regime, to 1, when we reach at the center the instability where the gas is going to collapse under the attractive atomic interaction. One can then calculate numerically exactly the mode frequencies as a function of λ [6]. For a given value of λ, one can then approximate the Hartree chemical potential by our model to obtain the parameters p and α through a least square fit. Then the mode frequencies are determined by Eq.(3). The results are given in Fig.1 for the monopole modes ($l = 0$) and are in close agreement with the numerical ones (the same is true for $l = 1$ and 2). This result shows that if we have , for a given λ, the frequencies of the first two modes, we can obtain the value of α and p, and therefore the chemical potential $\mu(n)$ for $0 < n < n(0)$. This shows that the measure of the modes can determine the equation of state $\mu(n)$.

One can also study the vicinity of collapse by using the quasipolynomial model, as is done in Ref.[6]. Morever, one could also fit the chemical potential by a quasipolynomial model described by e.g. the three parameters α, p_0 and p_1.

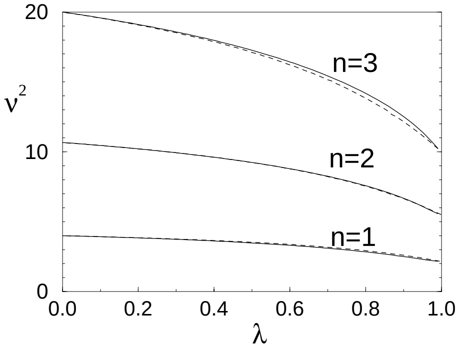

Fig. 1. Reduced frequency $\nu^2 = \omega^2/\Omega^2$ for a Fermi gas within the Hartree approximation as a function of the coupling constant λ. Full line: exact numerical solution. Dashed line: approximate analytical solution.

4. Conclusion

We have shown that the knowledge of the mode frequencies in trapped ultracold gases can in principle determine the chemical potential as a function of density, as we verified for the case of two equal fermionic populations in different hyperfine states with attractive interaction.

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