Stationary convection in a dilute rotating ^{3}He–superfluid ^{4}He mixture: a linear stability analysis

Michael S. Thurlow [a,1], Jayanta K. Bhattacharjee [b], Matthew J. Lees [a], Peter G. J. Lucas [a]

[a] *Department of Physics and Astronomy, University of Manchester, Manchester, M13 9PL, UK*

[b] *Indian Association for the Cultivation of Science, Calcutta 700032, India*

Abstract

A fluid layer subject to a vertical temperature gradient will start convecting when the temperature difference across the layer exceeds a critical value. A linear stability analysis is performed to calculate this critical value for a dilute ^{3}He–superfluid ^{4}He mixture rotating about a vertical axis. Significant differences from the result for a classical fluid are shown which arise from the presence of vortex mutual friction in the equations of motion. Unlike the non-rotating case these differences should be experimentally observable.

Key words: superfluid; convection; instability

1. Introduction

The addition of ^{3}He to liquid ^{4}He allows for a finite thermal conductivity in the superfluid phase and hence a temperature gradient can be supported. Consequently convective instability is possible in the mixture. The theory of onset of convection in these mixtures was effectively set up by Parshin [1]; more refined calculations of the onset of convection were carried out by Steinberg [2] and Fetter [3]. While these authors disagree on the issue of time-dependence, the two approaches yield identical answers with regard to stationary convection.

One of the interesting features of convection in superfluid mixtures is that the conduction state is one of dynamical equilibrium. Fetter showed that the effects brought about by the destabilization of the non-zero normal velocity are small and produce perturbative corrections to the Rayleigh number. When the perturbative effects are ignored, Fetter's analysis reduces to that of Parshin's. We want to investigate if rotation about a vertical axis substantially modifies the critical Rayleigh number for the onset of convection. For simplicity we will use Parshin's equations of motion augmented with the Coriolis and vortex mutual friction forces [4].

[1] Corresponding author. E-mail:mike.thurlow@man.ac.uk

2. Analysis

We consider a ^{3}He-^{4}He mixture contained between two horizontal thermally conducting boundaries of infinite extent and rotating about a vertical axis. We apply a temperature gradient across the fluid with the higher temperature on the top. This is in the opposite direction to that usually required to initiate convection — this is due to the ^{3}He heat flush [5]. Ignoring time dependencies, the equations governing the superfluid mixture dynamics (linearised and non-dimensionalised) are

$$\nabla^4 j = \nabla^4 q + R\nabla_{\rm h}^2\theta + 2\Omega\frac{\partial\zeta}{\partial z}, \quad (1)$$

$$\nabla^2\zeta = \nabla^2\zeta_{\rm s} - 2\Omega\frac{\partial j}{\partial z}, \quad (2)$$

$$\nabla^2\theta = j - q, \quad (3)$$

$$B\zeta_s = (2-B')\frac{\partial q}{\partial z} + B\frac{\rho_s}{\rho}\zeta + B'\frac{\rho_s}{\rho}\frac{\partial j}{\partial z}, \tag{4}$$

$$B\frac{\partial^2 q}{\partial z^2} = -(2-B')\frac{\partial \zeta_s}{\partial z} - B'\frac{\rho_s}{\rho}\frac{\partial \zeta}{\partial z} + B\frac{\rho_s}{\rho}\frac{\partial^2 j}{\partial z^2}. \tag{5}$$

Here j, q, ζ and ζ_s are perturbations from the conduction state values of the z-components of $\mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s$, $\mathbf{q} = \rho \mathbf{v}_s$, $\nabla\times\mathbf{j}$ and $\nabla\times\mathbf{q}$ respectively. θ is the temperature profile perturbation, R is the Rayleigh number, Ω is the angular velocity of rotation and B and B' are the mutual friction coefficients. ρ_n (ρ_s) and $\mathbf{v}_n$ ($\mathbf{v}_s$) are the normal (superfluid) density and velocity. ∇_h^2 is the horizontal Laplacian.

We can eliminate θ, q, ζ and ζ_s from equations (1) – (5) to obtain an expression for j. This variable has especially simple boundary conditions, namely $j = 0$ at the top and bottom boundaries. We find that

$$\begin{aligned} &\left(B_1 + \frac{\rho_s}{\rho}B_2\right)\nabla^2\left(\nabla^6 - R\nabla_h^2\right)\frac{\partial^2 j}{\partial z^2} \\ &-\frac{\rho_s}{\rho}\left(\frac{\rho_s}{\rho}B_3\nabla^2 - B_2\nabla^2 - 4\Omega B\right)\left(\nabla^6 - R\nabla_h^2\right)\frac{\partial^2 j}{\partial z^2} \\ &-\ 4\ \Omega\nabla^2\left(\Omega B_1 - B\nabla^2\frac{\rho_s}{\rho}\right)\frac{\partial^4 j}{\partial z^4} = 0 \end{aligned} \tag{6}$$

where $B_1 = B^2 + (2-B')^2$, $B_2 = B^2 - (2-B')B'$ and $B_3 = B^2 + B'^2$.

For the unbounded slab of fluid the solution to equation (6) is in the form of periodic structures in the horizontal x-y plane. The solution for j therefore has the form (for idealised stress-free boundaries)

$$j = \sin(\pi z)\exp[\mathrm{i}(k_x x + k_y y)]. \tag{7}$$

Substituting and rearranging we arrive at an expression for the Rayleigh number as a function of wavenumber

$$R = \frac{(\pi^2 + k^2)^3}{k^2} + \frac{4\pi^2\Omega^2}{k^2}\Phi \tag{8}$$

where

$$\Phi = \frac{B_1 + \frac{\rho_s}{\rho}(k^2+\pi^2)\frac{B}{\Omega}}{B_1 + 2\frac{\rho_s}{\rho}\left[\frac{2B\Omega}{k^2+\pi^2} - B_2\right] + \left(\frac{\rho_s}{\rho}\right)^2 B_3} \tag{9}$$

and $k^2 = k_x^2 + k_y^2$. Minimizing equation (9) with respect to k for given B, B', $\frac{\rho_s}{\rho}$ and Ω gives us the critical Rayleigh number and wavenumber for the onset of time-independent convection.

If the mutual friction terms are absent ($B = B' = 0$), then $\Phi = 1$ and equation (8) reduces to the form for a classical fluid. If we now include the mutual friction terms, we find significant deviation from the classical pure fluid result. For large Ω, we can easily see that $k_c(\Omega) \sim \Omega^{1/6}$ which leads to $R_c(\Omega) - R_c(0) \sim \Omega$. These exponents differ from the results for a pure fluid where they are $\frac{1}{3}$ and $\frac{4}{3}$ respectively [6].

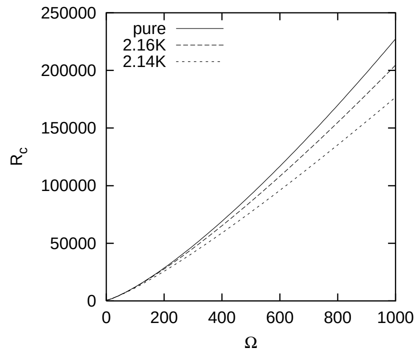

Fig. 1. Critical Rayleigh number as a function of Ω

Figure 1 shows the variation with angular velocity of the critical Rayleigh number for a pure fluid and a superfluid mixture at two temperatures. The values of B and B' for a superfluid mixture are not known but if we consider a dilute mixture near the λ-line but well above the phase-separated region we can expect that they will not differ too much from those of pure ^{4}He. We used $\frac{\rho_s}{\rho} = 0.09$, $B = 2.42$, $B' = -0.68$ at 2.16 K and $\frac{\rho_s}{\rho} = 0.16$, $B = 1.79$, $B' = -0.30$ at 2.14 K [7].

3. Conclusion

The departure from the standard value of R caused by B and B' has to be compared with the departure caused by the neglected two-fluid terms. Fetter estimates that the latter are too small to detect experimentally. The correction due to the mutual friction can be significant and we expect that it would be observable experimentally. However, at and above a finite value of Ω, the preferred onset state for convection is not stationary but oscillatory. Work is in progress to extend the above analysis to account for this time-dependence.

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